Optimal pure state cloning and optimal pure state transposition are complementary channels

Vanessa Brzić, Dmitry Grinko, Michał Studziński,
Marco Túlio Quintino

Sorbonne University, CNRS, LIP6

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- The rest is forbidden...

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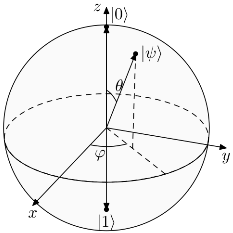
- Can we bypass the no-go theorem?
- How well can be approximate state cloning?

The universal NOT and state transpositon

- It would be useful negate qubits[1][2]: $|\psi\rangle \mapsto U|\psi\rangle = |\psi_\perp\rangle$



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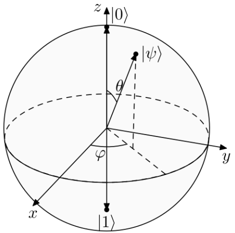

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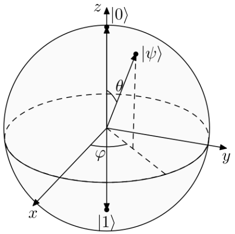

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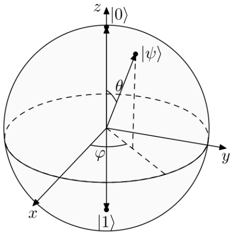

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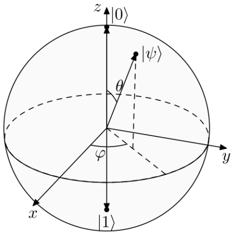

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- We come back to this before Result 1.

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Revisiting optimal cloning

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- Optimal aproxiations:

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- The black cow factor: $\max \eta$ such that

$$\mathrm{tr}_{\overline{O_i}}\left(\widetilde{C}\Big(|\psi\rangle\langle\psi|^{\otimes N} \Big)\right) = \eta\, |\psi\rangle\langle\psi| + (1-\eta)\frac{\mathbb{1}}{d}$$

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[**GM**] N. Gisin and S. Massar: "Optimal quantum cloning machines", Report `quant-ph/9705046`

[**GP**] N. Gisin and S. Popescu: "The Black Cow notes", unpublished

[**Hel**] C.W. Helstrom: *Quantum detection and estimation theory*, Academic Press, New York 1976

[**HB**] M. Hillery and V. Bužek: "Quantum copying: Fundamental inequalities",

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- Covariance 1: for all unitary $U \in SU(d)$,

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- Optimal $N \to (N+M)$ cloning: Bužek, Hillery (1996); Gisin, Massar (1997); Bruß, Ekert and Macchiavello (1998), Werner (1998); Keyl, Werner (1999).

$$\widetilde{C}(\rho_N) = \left(\frac{d_S^N}{d_S^{N+M}}\right) P_{N+M}^S \left(\rho_N \otimes \mathbb{1}^{\otimes M}\right) P_{N+M}^S + \dots$$

P_N^S is the projector onto the symmetric subspace and $d_S^N := \mathrm{tr}\left(P_N^S\right) = \binom{d+N-1}{N}$.

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- Black cow factor: $\eta = \frac{N}{N+M} \frac{(N+M+d)}{(N+d)}$

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- Optimal universal NOT $\implies$ for qubits[3], we have:

$$\max_{\widetilde{C}\in\mathrm{CPTP}} \int \mathrm{d}\psi\ F\Big(\widetilde{C}(|\psi\rangle\langle\psi|^{\otimes N}),\ (|\psi\rangle\langle\psi|^{T})\Big) = \frac{N+1}{N+2}$$

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where $d_S^N = \binom{d+N-1}{N}$ is the dimension of the symmetric subspace of N qudits.

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Theorem

$$\max_{\widetilde{C}\in\mathrm{CPTP}} \min_{|\psi\rangle} F\left(\widetilde{C}\left(|\psi\rangle\langle\psi|^{\otimes N}\right), |\psi\rangle\langle\psi|^{\mathsf{T}\otimes K}\right) = \frac{d_S^N}{d_S^{N+K}} \tag{1}$$

where $d_S^N := \dim(\mathrm{sym}^N(\mathbb{C}^d)) = \binom{N+d-1}{d-1}$.

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where $d_S^N := \dim(\mathrm{sym}^N(\mathbb{C}^d)) = \binom{N+d-1}{d-1}$.

When the domain of $\widetilde{C}$ *is restricted to the symmetric subspace, the solution of Eq. (1) is unique, and given by an estimation strategy, in particular, by* $\widetilde{T}_{\mathsf{CP}} : \mathcal{L}\left(\mathrm{sym}^N\left(\mathbb{C}^d\right)\right) \to \mathcal{L}(\mathbb{C}^d)^{\otimes K}$

$$\mathcal{T}_{\mathsf{CP}}(\rho) := d_S^N \int \mathrm{d}\psi \, \mathrm{Tr}\Big[\rho\, |\psi\rangle\langle\psi|^{\otimes N}\Big] \, |\psi\rangle\langle\psi|^{\mathsf{T}\otimes K} \qquad (2)$$

$$= \frac{d_S^N}{d_S^{N+K}} \mathrm{Tr}_{1\ldots N}\left[\Pi_{\mathsf{sym}}^{(N+K)}\left(\rho^{\mathsf{T}} \otimes \mathbb{1}_d^{\otimes K}\right)\right]. \qquad (3)$$

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for $M = 1$, $\eta^* = \frac{N}{N+d}$
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- For instance:

$$\widetilde{T}(\rho) = \frac{d_S^N}{d_S^{N+M}} \int \mathrm{d}\psi \;\; \mathrm{tr}\left(\rho \; |\psi\rangle\langle\psi|^{\otimes N}\right) \left(|\psi\rangle\langle\psi|^T\right)^{\otimes M} + \ldots$$

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- Present an ansatz for the dual problem (and use some basic rep theory)
- About Werner's optimality proof:
  "In any case, the difficulty of the optimality proofs should not hide the simplicity of the result"
  Quantum cloning V. Scarani, S. Iblisdir, N. Gisin, A. Acin, RMP (2005)

Proof method

Lemma

$$F*(d,N,M) = \max_{J \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_O)} \quad \mathrm{Tr}\left(J \frac{\Pi_{\mathrm{sym}}^{(N+K)}}{d_S^{N+K}} \right) \tag{4}$$

$$\text{s.t.} \quad J \geq 0, \quad \mathrm{Tr}_O(J) = I_I. \tag{5}$$

The dual SDP is

$$\min_{X \in \mathcal{L}(\mathbb{C}^d)^{\otimes N}} \quad \mathrm{Tr}(X) \tag{6}$$

$$\text{s.t.} \quad X \otimes I_{d^K} \geq \frac{\Pi_{\mathrm{sym}}^{(N+K)}}{d_S^{N+K}}. \tag{7}$$

The relationship between cloning transposition

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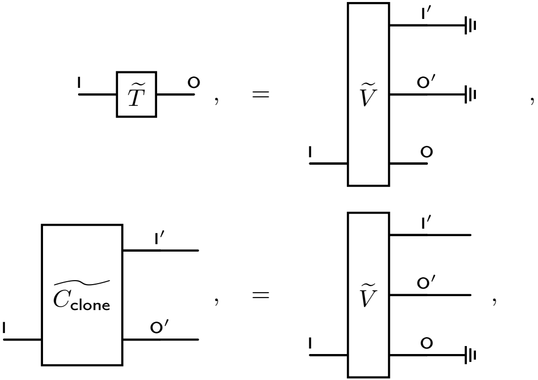
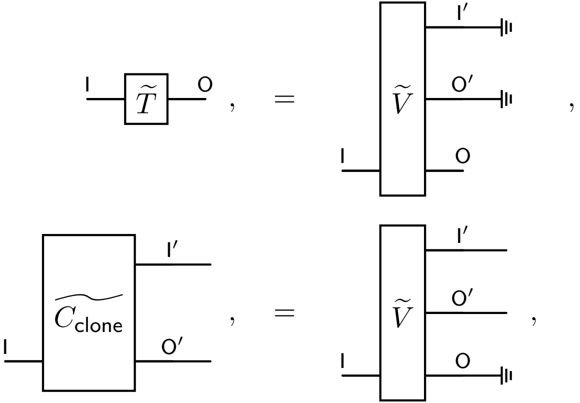
- $1 \to 2$ clone is complementary to $1 \to 1$ transposition
  V. Bužek and M. Hillery, PRA (1999),
  F. Buscemi, G. M. D'Ariano, P. Perinotti, M. F. Sacchi (PLA, 2003)



Cloning *Vs* Transposition

- Result 2: For any $d, N, M \in \mathbb{N}$,
  optimal (pure) $N \to (N+M)$ clone is a complementary channel of optimal (pure) $N \to M$ transposition.



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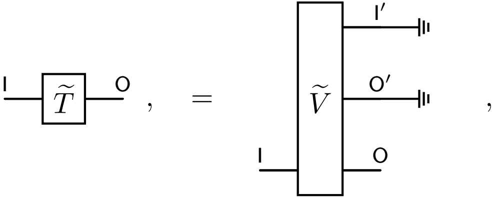
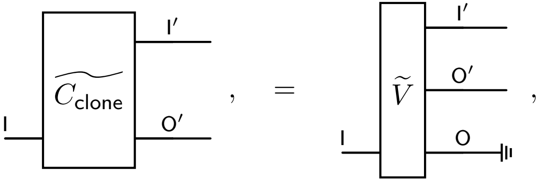
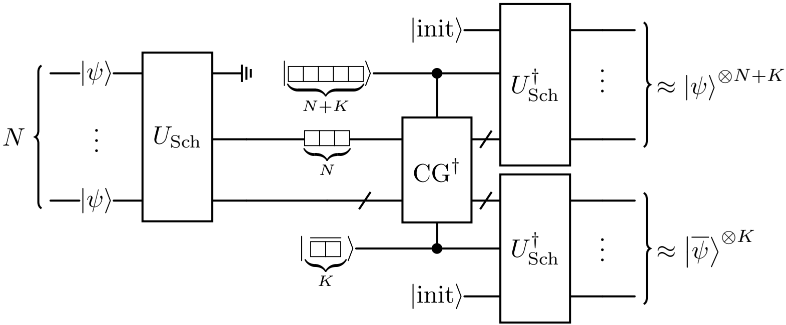




- Cloning and transposition are attained by the same machine!

Cloning *Vs* Transposition



Inspired by:
Classification and implementation of unitary-equivariant and permutation-invariant quantum channels
Laura Mančinska, Elias Theil
arXiv:2510.08154

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- Q: The complementary channel is not unique, right?
- A: Indeed, there is an isometric freedom
- This freedom decides the basis of the transposition, and the "reference frame"

Degradable channels

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- Therefore, the optimal cloning channel is degradable.

Degradable channels

- Corollary: The optimal $N \to N+M$ cloning channel $\widetilde{C}_{\mathrm{clone}}$ is degradable, with

$$\widetilde{T}_{\mathrm{CP}}^{(N\to M)} = \widetilde{T}_{\mathrm{CP}}^{(N+M\to M)} \circ \widetilde{C}_{\mathrm{clone}}.$$

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How about mixed states?

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$$\eta^*(d,N) = \begin{cases} \frac{1}{d+1} & \text{if } N \leq d-1, \\ \frac{N}{d(d-1)+N} & \text{if } N \geq d-1. \end{cases}$$

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- For $d = 2$, we always have $\rho = v\,|\psi\rangle\langle\psi| + (1-v)\frac{\mathbb{1}}{2}$.

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- A: Solve a minimal eigenvalue problem:
  Implementing positive maps with multiple copies of an input state
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Transposition on density matrices

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  Implementing positive maps with multiple copies of an input state
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- The minimal eigenvalue of $T_N := \left(\sum_{i=1}^{N} F_{0i} \otimes \mathbb{1}_{\bar{i}}\right)$ can be calculated (Jucys–Murphy elements)

Applications?

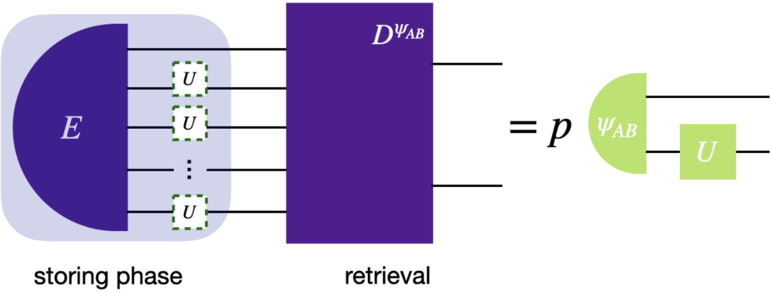
The universal NOT with many copies is useful to correct the unitary storage-and-retrieval:

Quantum Physics

[Submitted on 23 Oct 2025 (v1), last revised 16 Nov 2025 (this version, v2)]

Higher-order quantum computing with known input states

Vanessa Brzić, Satoshi Yoshida, Mio Murao, Marco Túlio Quintino



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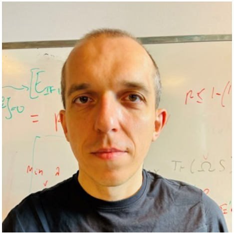
Thank you to my co-authors!

Vanessa Brzić

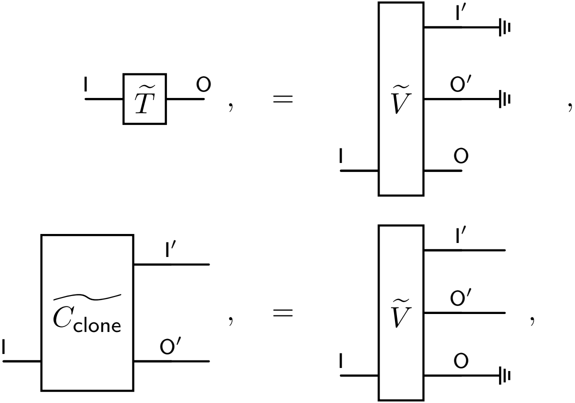
Dmitry Grinko



Michał Studziński

Thank you to my group!

Thank you!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