Optimal pure state cloning and optimal pure state transposition are complementary channels

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- The rest is forbidden. . .

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- How well can be approximate state cloning?

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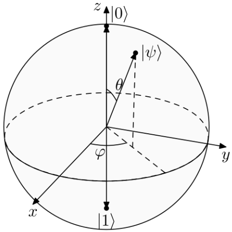

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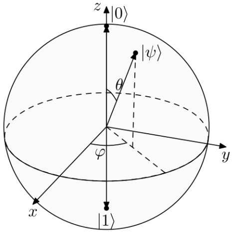

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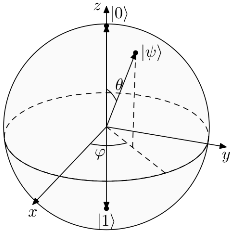

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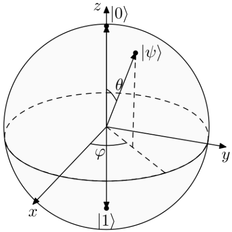

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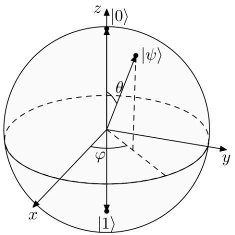

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- We come back to this before Result 1.

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Revisiting optimal cloning

Optimal quantum cloning

- Optimal aproxiations:

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- The black cow factor: $\max \eta$ such that

$$\mathrm{tr}_{\overline{O_i}}\Big(\widetilde{C}\Big(\,|\psi\rangle\langle\psi|^{\otimes N}\,\Big)\Big) = \eta\,|\psi\rangle\langle\psi| + (1-\eta)\frac{\mathbb{1}}{d}$$

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* The reason for this terminology is that it plays an important role in discussions of the cloning problem started by Chiara Machiavello and Artur Ekert at the Black Cow Café in Croton-on-Hudson, NY, and further clarified in collaboration with Dagmar Bruß [**BEM**]. I learned about this line of argument from a set of "Black Cow Notes" by Nicolas Gisin and Sandu Popescu.

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[**GM**] N. Gisin and S. Massar: "Optimal quantum cloning machines", Report `quant-ph/9705046`

[**GP**] N. Gisin and S. Popescu: "The Black Cow notes", unpublished

[**Hel**] C.W. Helstrom: *Quantum detection and estimation theory*, Academic Press, New York 1976

[**HB**] M. Hillery and V. Bužek: "Quantum copying: Fundamental inequalities",

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- Optimal $N \to (N+M)$ cloning: Bužek, Hillery (1996); Gisin, Massar (1997); Bruß, Ekert and Macchiavello (1998), Werner (1998); Keyl, Werner (1999).

$$\widetilde{C}(\rho_N) = \left(\frac{d_S^N}{d_S^{N+M}}\right) P_{N+M}^S \left(\rho_N \otimes \mathbb{1}^{\otimes M}\right) P_{N+M}^S + \dots$$

P_N^S is the projector onto the symmetric subspace and $d_S^N := \mathrm{tr}\left(P_N^S\right) = \binom{d+N-1}{N}$.

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- Black cow factor: $\eta = \frac{N}{N+M} \frac{(N+M+d)}{(N+d)}$

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where $d_S^N = \binom{d+N-1}{N}$ is the dimension of the symmetric subspace of N qudits.

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- For $M = 1$,

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for $M = 1$, $\eta^* = \frac{N}{N+d}$
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- Present an ansatz for the dual problem (and use some basic rep theory)
- About Werner's optimality proof:
  "In any case, the difficulty of the optimality proofs should not hide the simplicity of the result"
  Quantum cloning V. Scarani, S. Iblisdir, N. Gisin, A. Acin, RMP (2005)

The relationship between cloning transposition

Cloning *Vs* Transposition

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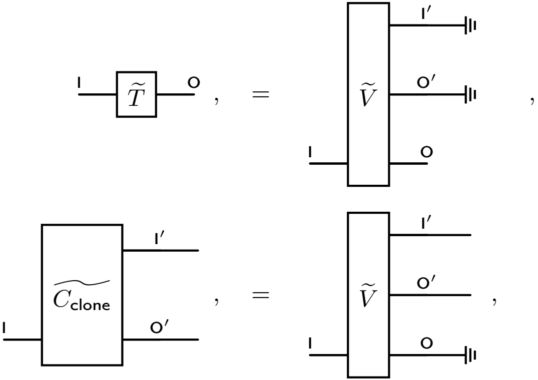
- $1 \to 2$ clone is complementary to $1 \to 1$ transposition
  V. Bužek and M. Hillery, PRA (1999),
  F. Buscemi, G. M. D'Ariano, P. Perinotti, M. F. Sacchi (PLA, 2003)



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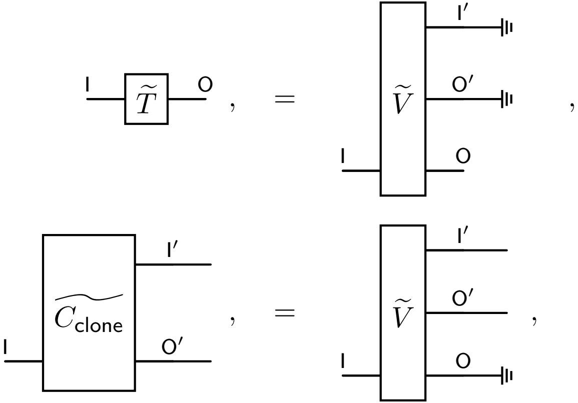
- Result 2: For any $d, N, M \in \mathbb{N}$, optimal (pure) $N \to (N+M)$ clone is a complementary channel of optimal (pure) $N \to M$ transposition.



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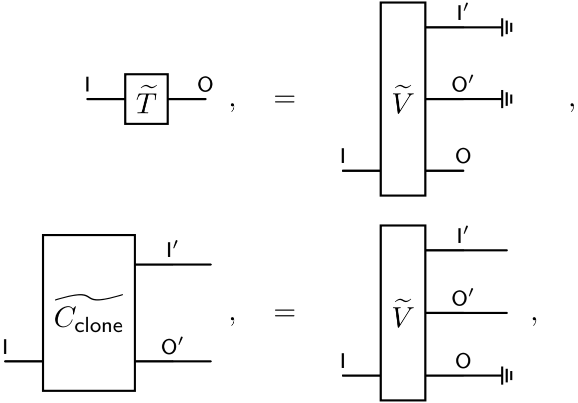

- Cloning and transposition are attained by the same machine!

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Cloning *Vs* Transposition

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- This freedom decides the basis of the transposition, and the "reference frame"

How about mixed states?

Transposition on density matrices

- Random density matrices: Hilbert-Schmidt measure, Bures measure, ...

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- $F(\rho, \sigma) := \mathrm{tr}\big(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\big)^2$, not linear
- How about covariance and white noise visibility?

$$\eta^*(d, N) := \max_{\widetilde{C} \in \mathsf{CPTP}} \eta$$

$$\widetilde{C}\left(\rho^{\otimes N}\right) = \eta \rho^T + (1-\eta)\frac{\mathbb{1}}{d}, \quad \forall \rho$$

Transposition on density matrices

- Result 3: For any $d, N \in \mathbb{N}$,

$$\eta^*(d,N) = \begin{cases} \frac{1}{d+1} & \text{if } N \leq d-1, \\ \frac{N}{d(d-1)+N} & \text{if } N \geq d-1. \end{cases}$$

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- For pure states (Result 1): $\eta^*_\psi(d,N) = \frac{N}{N+d} \ge \eta^*(d,N)$, why?
- For $N > 1$, span$\left(|\psi\rangle\langle\psi|^{\otimes N}\right) \neq$ span$\left(\rho^{\otimes N}\right)$
- For $d = 2$, we always have $\rho = v\,|\psi\rangle\langle\psi| + (1-v)\frac{\mathbb{1}}{2}$.

Transposition on density matrices

- Q: How did you prove result 3?

Transposition on density matrices

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- A: Solve a minimal eigenvalue problem:
  Implementing positive maps with multiple copies of an input state
  Q. Dong, M.T. Quintino, A. Soeda, and M. Murao, PRA (2019)

Transposition on density matrices

- Q: How did you prove result 3?
- A: Solve a minimal eigenvalue problem:
  Implementing positive maps with multiple copies of an input state
  Q. Dong, M.T. Quintino, A. Soeda, and M. Murao, PRA (2019)
- The minimal eigenvalue of $T_N := \left(\sum_{i=1}^{N} F_{0i} \otimes \mathbb{1}_{\bar{i}}\right)$ can be calculated (Jucys–Murphy elements)

Applications?

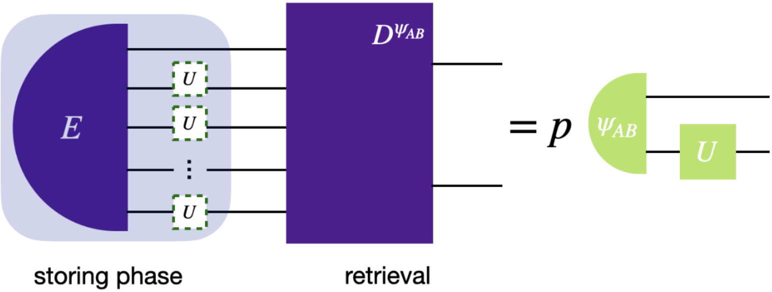
The universal NOT with many copies is useful to correct the unitary storage-and-retrieval:

Quantum Physics

[Submitted on 23 Oct 2025 (v1), last revised 16 Nov 2025 (this version, v2)]

Higher-order quantum computing with known input states

Vanessa Brzić, Satoshi Yoshida, Mio Murao, Marco Túlio Quintino



Conclusions

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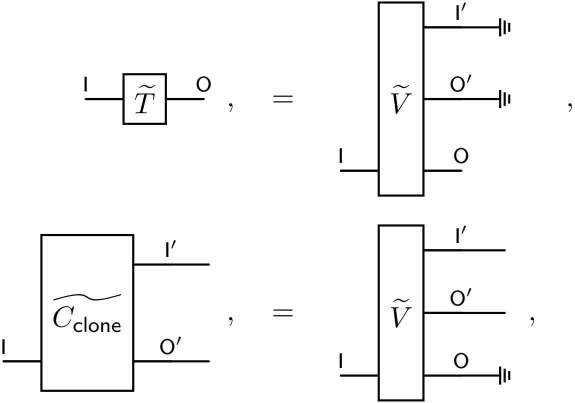
Thank you!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