One-to-one correspondence between deterministic port-based teleportation and unitary estimation

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QPL: July 17, 2025

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Four equivalent problems

- Unitary estimation

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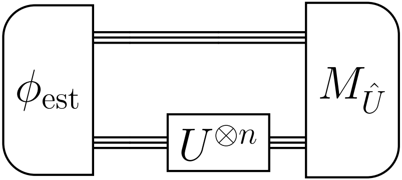
- Deterministic unitary storage-and-retrieval (dSAR)
- Parallel unitary inversion/ parallel unitary transposition (Inv_{par})

Unitary Estimation (Est)



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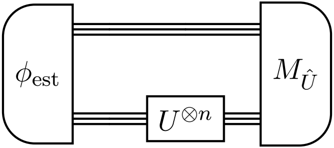

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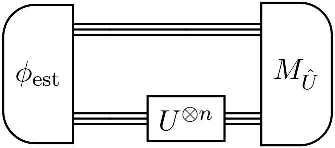

- Fidelity for each U: $\langle F(U) \rangle = \int_{\hat{U}} p(\hat{U}|U)\; F(U, \hat{U})$
- For a probe state ϕ_{est} and POVM $\{M_{\hat{U}}\}_{\hat{U}}$:

$$p(\hat{U}|U) := \operatorname{tr}\left(\left(U^{\otimes k} \otimes \mathbb{1} \right) \phi_{\text{est}} \left(U^{\otimes k} \otimes \mathbb{1} \right)^{\dagger} \; M_{\hat{U}} \right)$$

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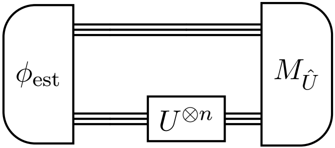

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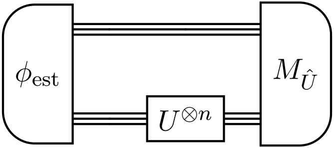

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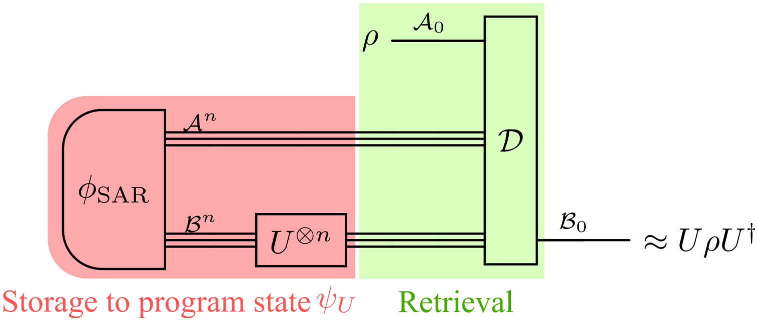
Deterministic Storage-and-Retrieval (dSAR)



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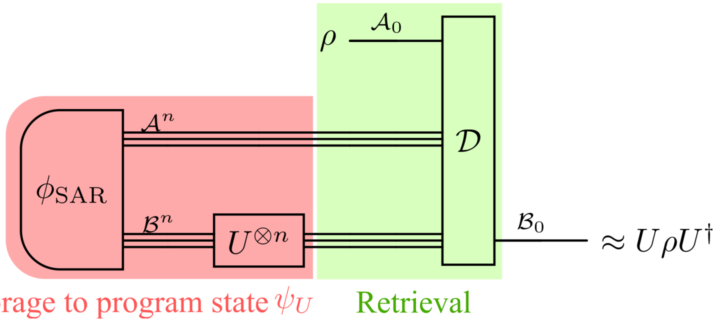

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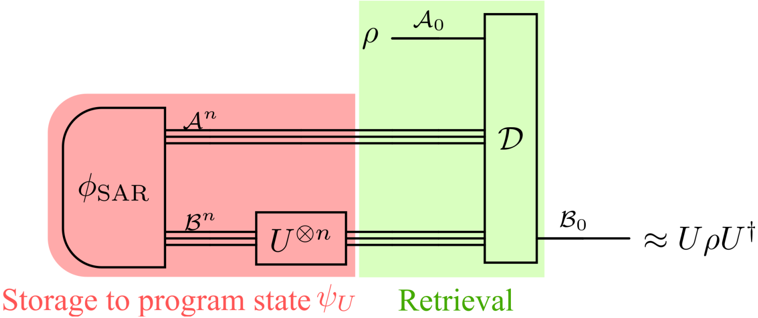

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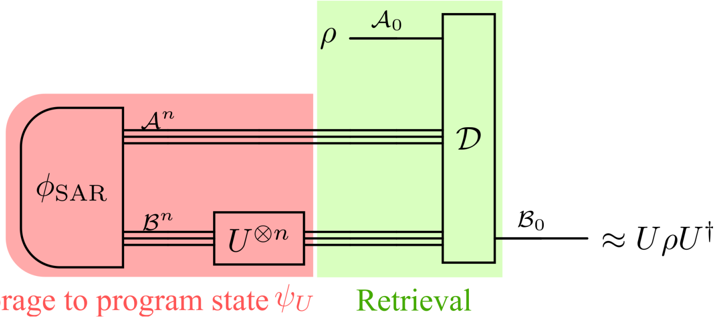

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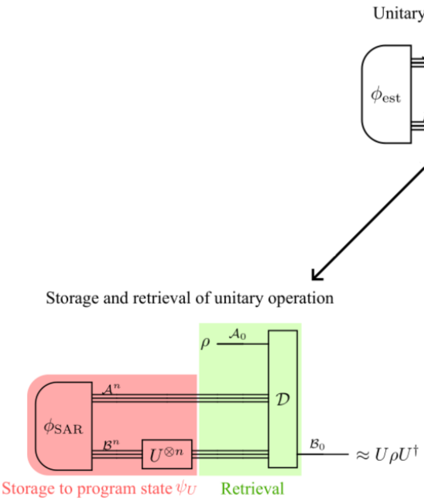
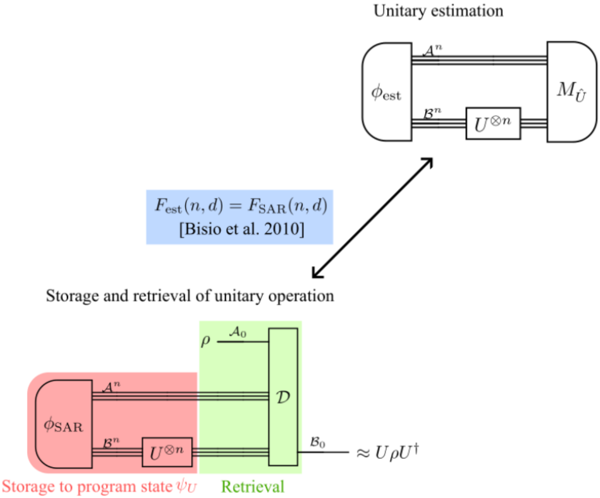
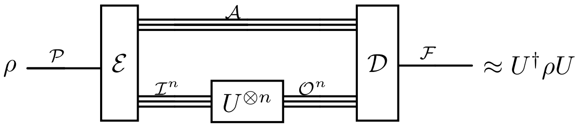
Est $\Longrightarrow$ dSAR



dSAR $\Longrightarrow$ Est



Deterministic parallel unitary inversion ($\mathsf{Inv}_{\mathsf{par}}$)



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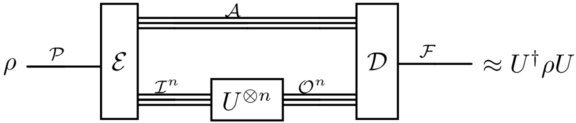

- For an encoder channel $\mathcal{E}$ and decoder channel $\mathcal{D}$: $\mathcal{C} := \mathcal{D} \circ (\mathcal{I} \otimes \mathcal{U})^{\otimes n} \circ \mathcal{E}$

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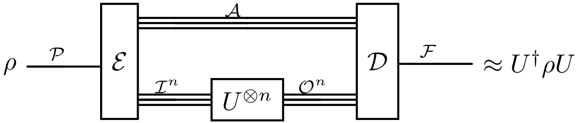

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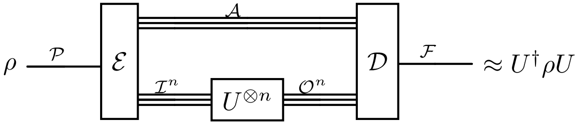

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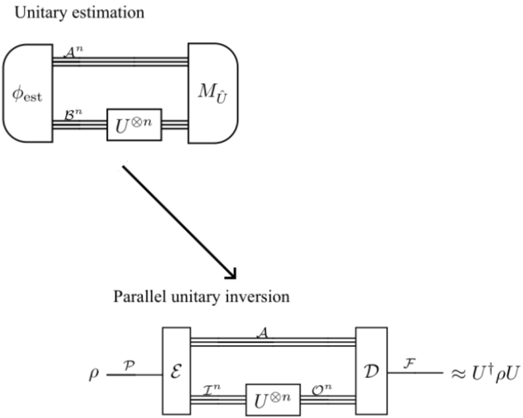
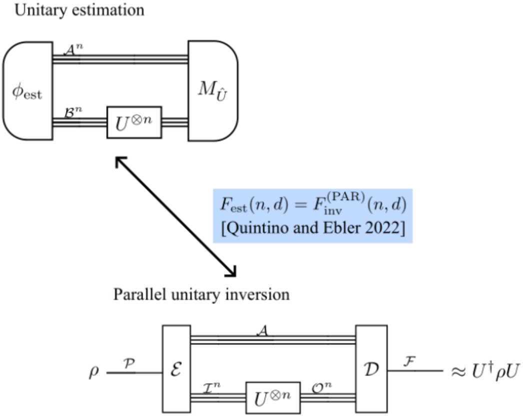
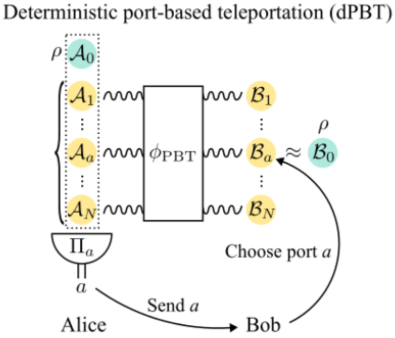


$\mathrm{Inv}_{\mathrm{par}} \implies \mathrm{Est}$



Deterministic Port-Based Teleportation(dPBT)



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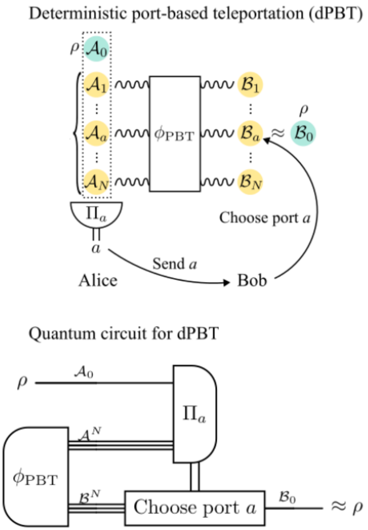

dPBT $\implies$ dSAR

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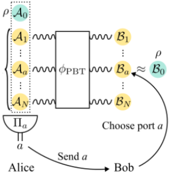
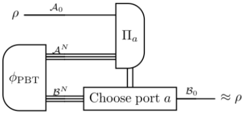
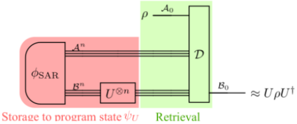


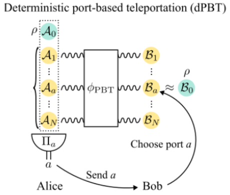
Quantum circuit for dPBT

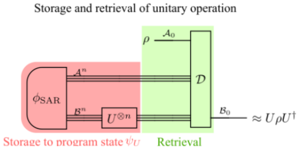


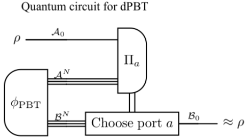
Storage and retrieval of unitary operation



dSAR $\implies$ dPBT?







Does the converse hold?

dSAR $\implies$ dPBT?

For the probabilistic case, yes.

dSAR $\implies$ dPBT?

For the probabilistic case, yes.

Michał Studziński, Sergii Strelchuk, Marek Mozrzymas, Michał Horodecki
Port-based teleportation in arbitrary dimension, Sci Rep. (2017)

M. Sedlák, A. Bisio, and M. Ziman,
Optimal Probabilistic Storage and Retrieval of Unitary Channels, PRL (2019)

$$p_{\mathsf{PBT}}(d, N) = p_{\mathsf{SAR}}(d, N) = 1 - \frac{d^2 - 1}{N + d^2 - 1}$$

dSAR $\Longrightarrow$ dPBT?

Main result

Main result:

Given an n-call unitary estimation with fidelity $F(n, d)$,
there exists an $n + 1$ ports dPBT with fidelity $F(n, d)$, *vice versa*

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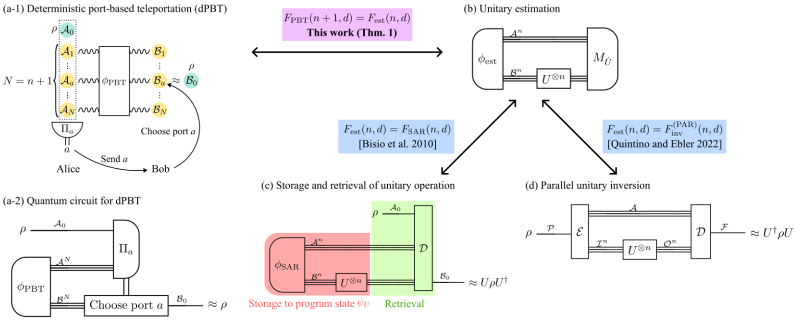
The proof is constructive (and covariant).

Main result



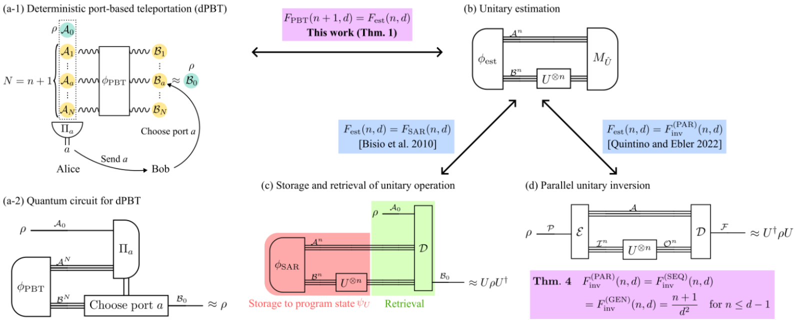
(a-1) Deterministic port-based teleportation (dPBT)
$N = n+1$
ϕ_{PBT}
Π_a
Choose port a
Send a
Alice
Bob
$F_{\mathrm{PBT}}(n+1,d) = F_{\mathrm{est}}(n,d)$
This work (Thm. 1)
(b) Unitary estimation
ϕ_{est}
$U^{\otimes n}$
$M_{\hat{U}}$
$F_{\mathrm{est}}(n,d) = F_{\mathrm{SAR}}(n,d)$
[Bisio et al. 2010]
$F_{\mathrm{est}}(n,d) = F_{\mathrm{inv}}^{(\mathrm{PAR})}(n,d)$
[Quintino and Ebler 2022]
(a-2) Quantum circuit for dPBT
Choose port a
$\approx \rho$
(c) Storage and retrieval of unitary operation
ϕ_{SAR}
$\approx U\rho U^\dagger$
Storage to program state ψ_U
Retrieval
(d) Parallel unitary inversion
$\approx U^\dagger \rho U$

$n \leq d-1$

Bonus result:

For $n \leq d-1$ calls,
$F_{\text{inv}}^{\text{PAR}}(n,d) = F_{\text{inv}}^{\text{SEQ}}(n,d) = F_{\text{inv}}^{\text{GEN}}(n,d) = \frac{n+1}{d^2}$.

$n \leq d - 1$



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Thm. 4 $F_{\rm inv}^{\rm (PAR)}(n,d) = F_{\rm inv}^{\rm (SEQ)}(n,d)$
$= F_{\rm inv}^{\rm (GEN)}(n,d) = \frac{n+1}{d^2}$ for $n \leq d-1$

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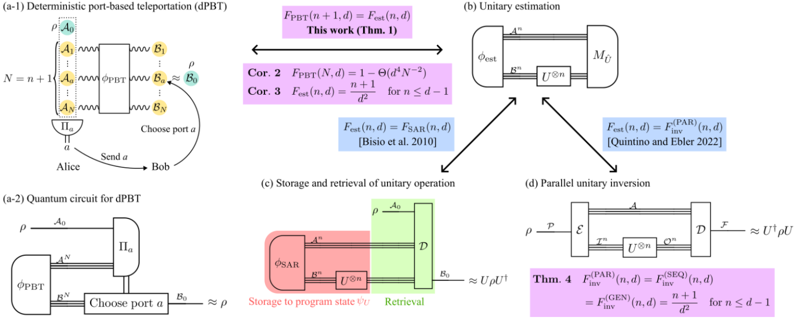
$$F_{\mathsf{Est}}(n,d) \leq 1 - \Omega\left(\frac{d^4}{n^2}\right)$$

- It follows from our main result that:

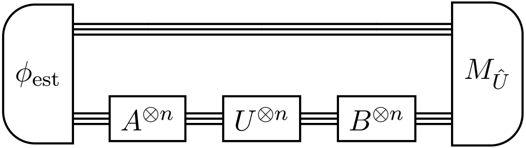
$$F_{\mathsf{PBT}}(n,d) = 1 - \Theta\left(\frac{d^4}{n^2}\right)$$

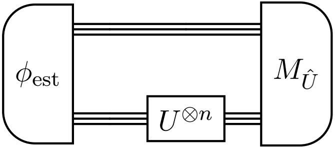
Main results



Methods:

- In Est, we can assume covariance w.l.g., $\forall A, B, \in \mathsf{SU}(d)$





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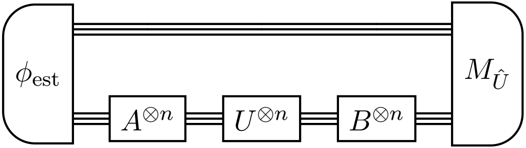

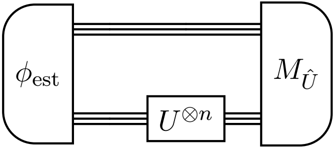

- Performance of covariant protocols:

$$F_{\mathsf{Est}}(n,d) = \langle s| \, M_{\mathsf{Est}}(n,d) \, |s\rangle$$

E. Bagan, M. Baig, and R. Muñoz-Tapia, PRA (2004)
G. Chiribella, G. M. D'Ariano, M. F. Sacchi, PRA (2005)

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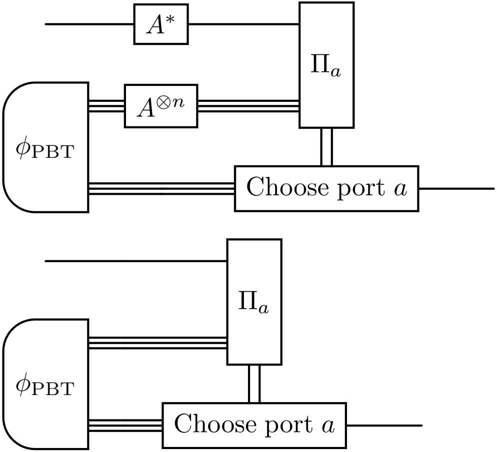

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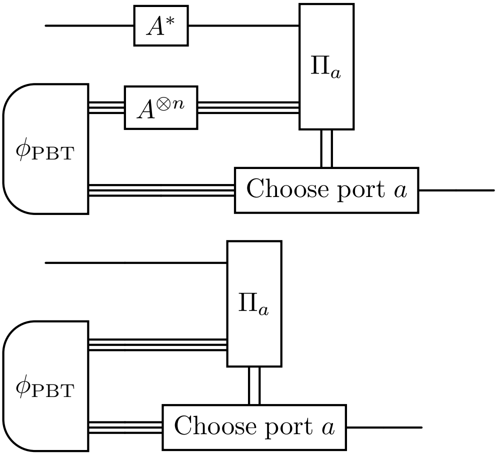

- M. Mozrzymas at al. NJP (2018) :

$$F_{\mathsf{PBT}}(n,d) = \langle v | M_{\mathsf{PBT}}(n,d) | v \rangle$$

Methods:

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- Since $|s\rangle\langle s| \leq \mathbb{1}$, it holds that

$$\langle s| M_{\mathsf{Est}}(n,d) |s\rangle \leq \langle v| M_{\mathsf{PBT}}(n+1,d) |v\rangle$$

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- "A retangular matrix decomposition"
  M. Mozrzymas at al. NJP (2018)

$$M_{\mathsf{PBT}}(n+1,d) = R(n,d)^T R(n,d)$$

- We show that

$$M_{\mathsf{Est}}(n,d) = R(n,d)R(n,d)^T$$

- If $|v\rangle$ is a strategy for PBT$(n+1,d)$, define $|s\rangle := \frac{R(n,d)|v\rangle}{\|R(n,d)|v\rangle\|}$, a valid Est$(n,d)$ strategy.
- Since $|s\rangle\langle s| \leq \mathbb{1}$, it holds that

$$\langle s| M_{\mathsf{Est}}(n,d) |s\rangle \leq \langle v| M_{\mathsf{PBT}}(n+1,d) |v\rangle$$

- Analogously,

$$\langle s| M_{\mathsf{Est}}(n,d) |s\rangle \geq \langle v| M_{\mathsf{PBT}}(n+1,d) |v\rangle$$

Final remarks

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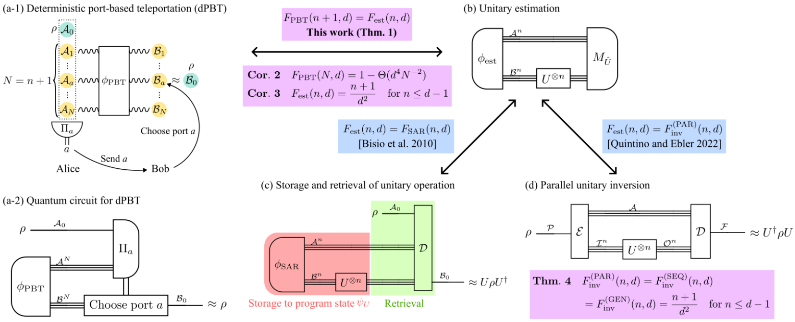
- This correspondence is explicit, and covariant by construction.
- Covariant protocol may require more resources than non-covariant one (e.g. larger auxiliary space (larger memory))
- How does Est and dPBT relate if we consider such resources?
- *Why*? $F_{\mathsf{Est}}(n,d) = F_{\mathsf{dPBT}}(n+1,d)$

Thank you!



(a-1) Deterministic port-based teleportation (dPBT)
$N = n+1$
ϕ_{PBT}
Π_a
Choose port a
Send a
Alice
Bob
$F_{\mathrm{PBT}}(n+1,d) = F_{\mathrm{est}}(n,d)$
This work (Thm. 1)
Cor. 2 $F_{\mathrm{PBT}}(N,d) = 1 - \Theta(d^4 N^{-2})$
Cor. 3 $F_{\mathrm{est}}(n,d) = \frac{n+1}{d^2}$ for $n \leq d-1$
(b) Unitary estimation
ϕ_{est}
$U^{\otimes n}$
$M_{\hat{U}}$
$F_{\mathrm{est}}(n,d) = F_{\mathrm{SAR}}(n,d)$
[Bisio et al. 2010]
$F_{\mathrm{est}}(n,d) = F_{\mathrm{inv}}^{(\mathrm{PAR})}(n,d)$
[Quintino and Ebler 2022]
(a-2) Quantum circuit for dPBT
Choose port a
$\approx \rho$
(c) Storage and retrieval of unitary operation
ϕ_{SAR}
$\approx U\rho U^\dagger$
Storage to program state ψ_U
Retrieval
(d) Parallel unitary inversion
$\approx U^\dagger \rho U$
Thm. 4 $F_{\mathrm{inv}}^{(\mathrm{PAR})}(n,d) = F_{\mathrm{inv}}^{(\mathrm{SEQ})}(n,d)$
$= F_{\mathrm{inv}}^{(\mathrm{GEN})}(n,d) = \frac{n+1}{d^2}$ for $n \leq d-1$