The relationship between measurement incompatibility and Bell nonlocality

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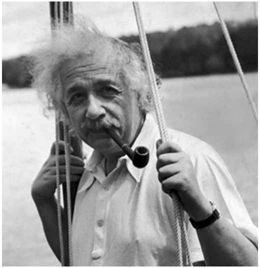
Measurement incompatibility

$$\Delta x \, \Delta p \geq \hbar/2$$

Quantum entanglement

$$\rho_{AB} \neq \int \pi(\lambda) \rho_A^\lambda \otimes \rho_B^\lambda \, \mathrm{d}\lambda$$



State+Measurement

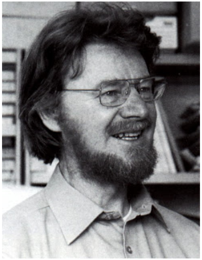
$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i)$$

State+Measurement

$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i), \quad |\langle i|\psi\rangle|^2$$

Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$



Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

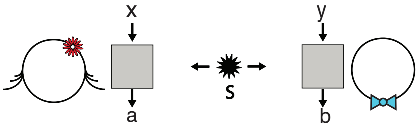
Bell NL $\implies$ Entanglement + Measurement incompatibility

Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

$$\text{Bell NL} \implies \text{Entanglement} + \text{Measurement incompatibility}$$
$$\text{Bell NL} \overset{?}{\impliedby} \text{Entanglement} + \text{Measurement incompatibility}$$

Bell Nonlocality



The Correlation/Anticorrelation Game

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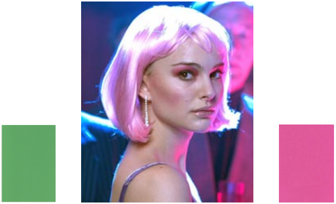
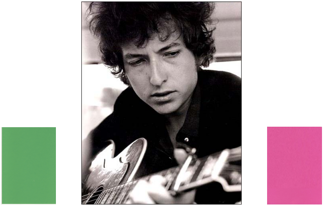
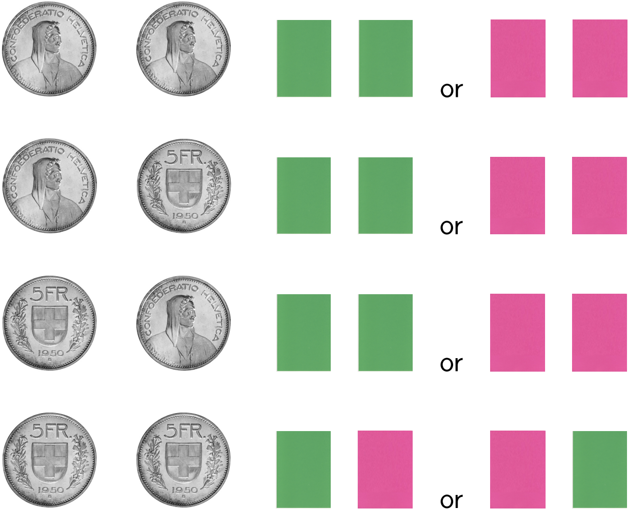




The Correlation/Anticorrelation Game

Winning Conditions



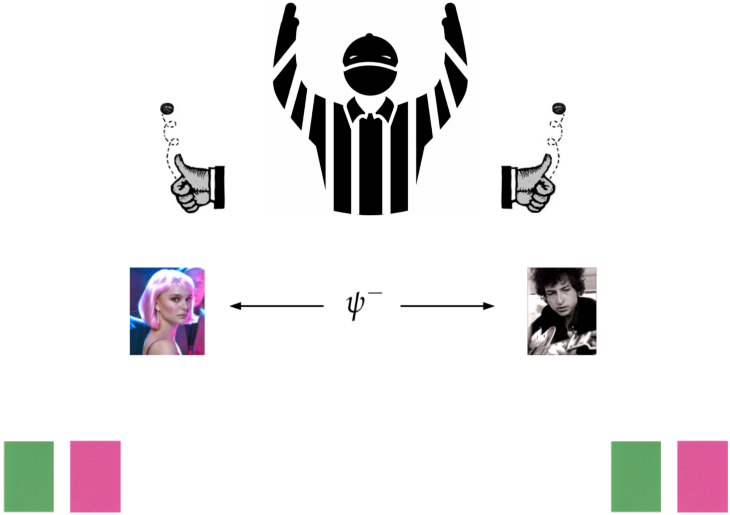
Best Strategy

Can Alice and Bob always win?

Best Strategy

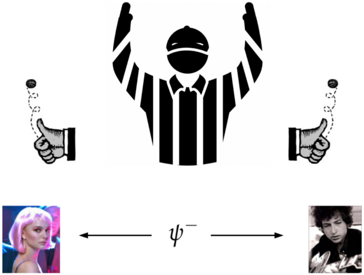
Best (classical) strategy wins with probability $\frac{3}{4}$

Quantum Strategy



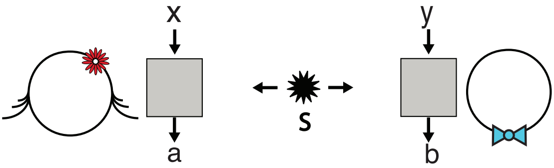
Quantum Strategy

$$p_q = \frac{2+\sqrt{2}}{4} \approx 0.8535$$



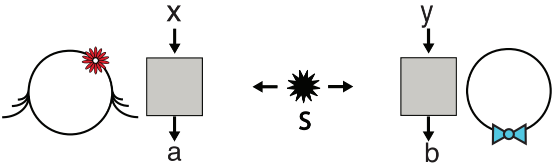
Bell nonlocality

$$p_{\text{win}} = \sum_{abxy} \pi(x,y) V(ab|xy) p(ab|xy)$$



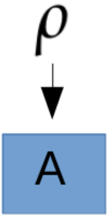
Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

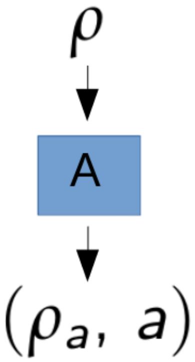


Quantum measurement

Quantum measurement



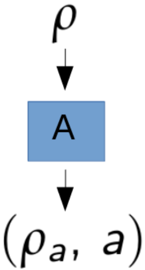
Quantum measurement



Quantum measurement: POVM

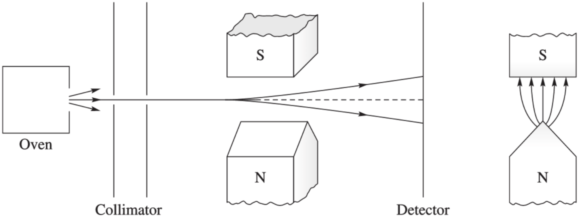
$$p(a|\rho, A) = \mathrm{tr}(\rho A_a)$$

$$A = \{A_a\}, \quad A_a \geq 0, \quad \sum_a A_a = I$$



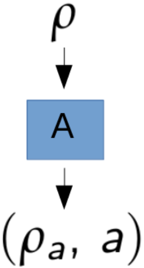
Quantum measurement: Naimark dilation

POVM $\iff$ global unitary + measurement



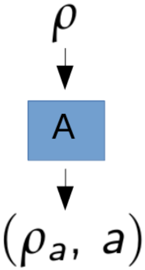
Quantum measurement

$$p(a|\rho, A) = \mathrm{tr}(\rho A_a), \quad \rho_a = A_a \text{ (if projective)}$$

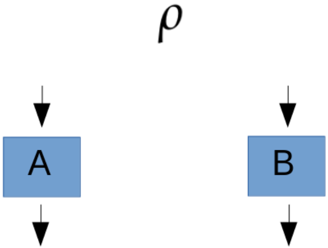


Quantum measurement

$$p(a|\rho, A) = \mathrm{tr}(\rho A_a), \quad \rho_a = \frac{\sqrt{A_a}\rho\sqrt{A_a}}{\mathrm{tr}(\rho A_a)} \text{ (Lüders)}$$

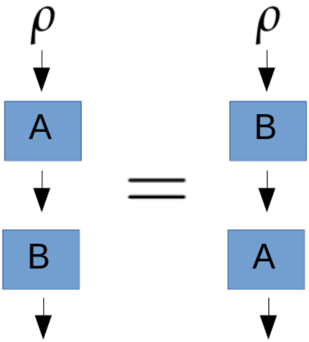


Measurement compatibility



Measurement compatibility

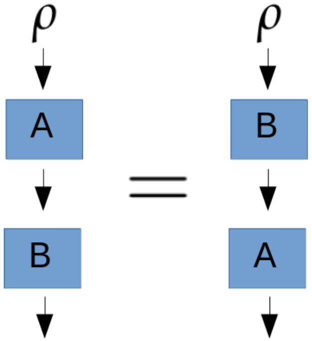
When the measurements commute:



Measurement compatibility

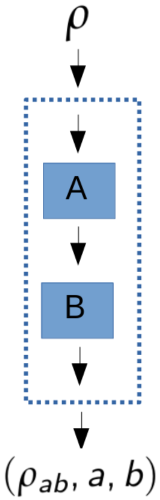
When the measurements commute:

$$[A_a, B_b] := A_a B_b - B_b A_a = 0$$



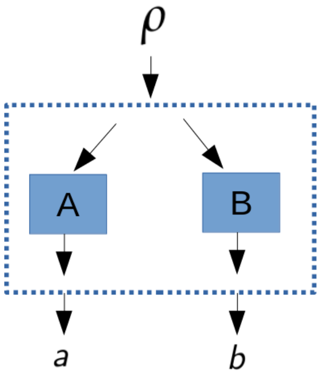
Measurement compatibility

Commutation $\implies$ measurement compatibility



Measurement compatibility

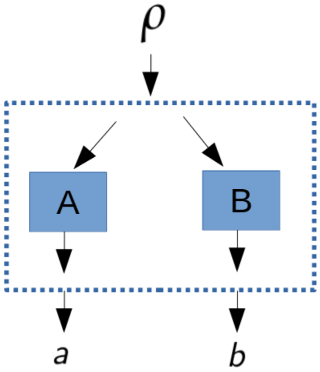
Joint measurability



Joint Measurability

$\{A_a\}$ and $\{B_b\}$ are JM if there exists a measurement $\{M_{ab}\}$ s. t.:

$$\sum_a M_{ab} = B_b$$
$$\sum_b M_{ab} = A_a$$



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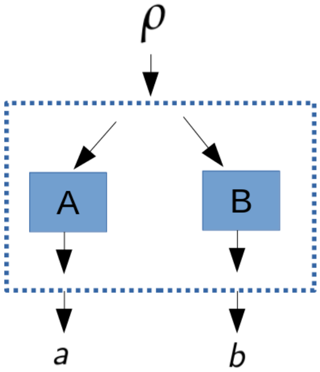

Pauli Measurements

$$\sigma_Z : \{|0\rangle\langle 0|,\ |1\rangle\langle 1|\} \quad \sigma_X : \{|+\rangle\langle +|,\ |-\rangle\langle -|\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2}\,; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

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$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Joint Measurability}$$

Hollow Triangle

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2} \; ; \quad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

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$$\sigma_{Y,\eta} : \left\{ \eta|Y+\rangle\langle Y+| + (1-\eta)\frac{I}{2} \; ; \quad \eta|Y-\rangle\langle Y-| + (1-\eta)\frac{I}{2} \right\}$$

$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Pairwise Measurability}$$

Hollow Triangle

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2} \; ; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

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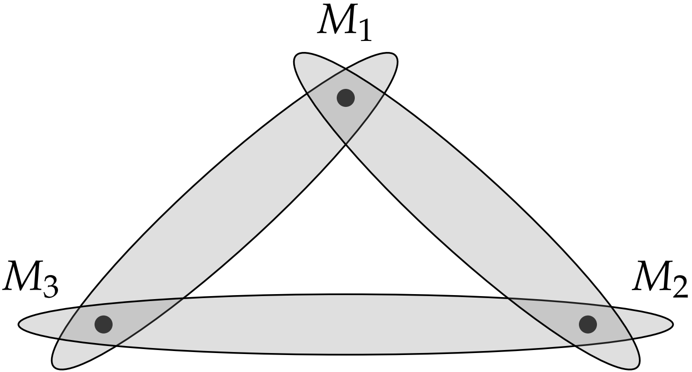
$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Pairwise Measurability}$$

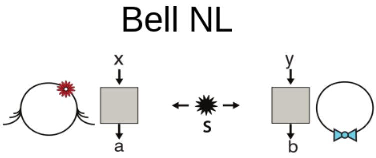
$$\eta \leq \frac{1}{\sqrt{3}} \implies \text{Triplewise Measurability}$$

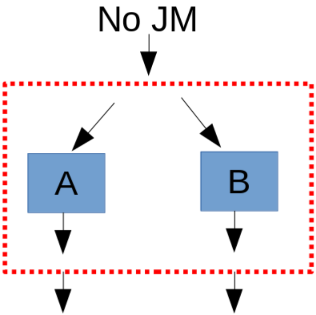
Hollow Triangle



T. Heinosaari, D. Reitzner, P. Stano: Foundations of Physics (2008)

Bell NL and no JM





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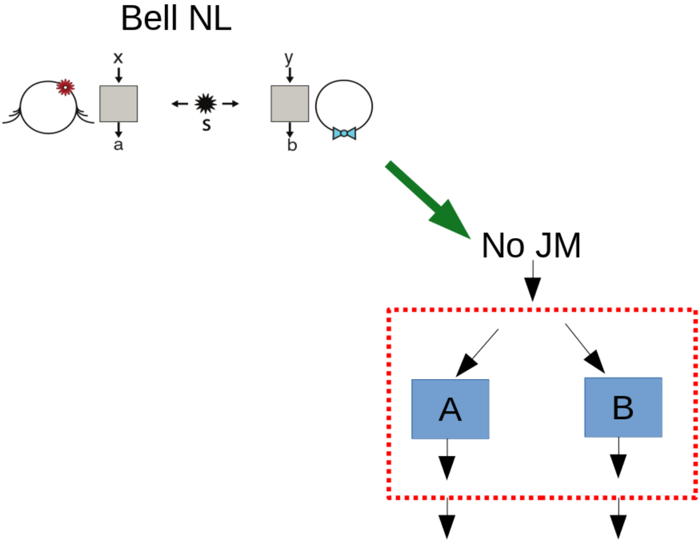
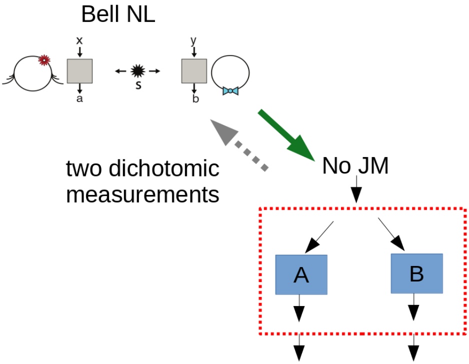


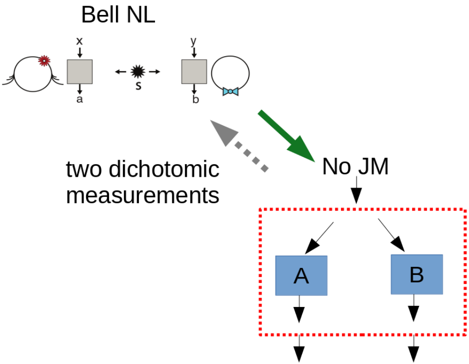
Bell NL and JM



M. Wolf, D. Perez-Garcia, C. Fernandez, PRL (2009)

Bell NL and JM

$\{A_{a|x}\}_{a,x=1}^{2}$ not JM $\implies$ $\exists \rho_{AB}$ and $\{B_{b|y}\}$ such that:
$p(ab|xy) = \mathrm{tr}(\rho_{AB}\, A_{a|x} \otimes B_{b|y})$ is Bell NL



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Bell nonlocality

- Are all incompatible measurements useful for Bell NL?

Bell nonlocality

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- All incompatible measurements useful for EPR steering!
  Joint measurability, EPR steering, and Bell nonlocality
  MT. Quintino, T. Vértesi, N. Brunner, PRL (2015)

  One-to-one mapping between steering and joint measurability problems
  R. Uola, C. Budroni, O. Gühne, JP. Pellonpää, PRL (2015)

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- How to prove or disprove such a thing?

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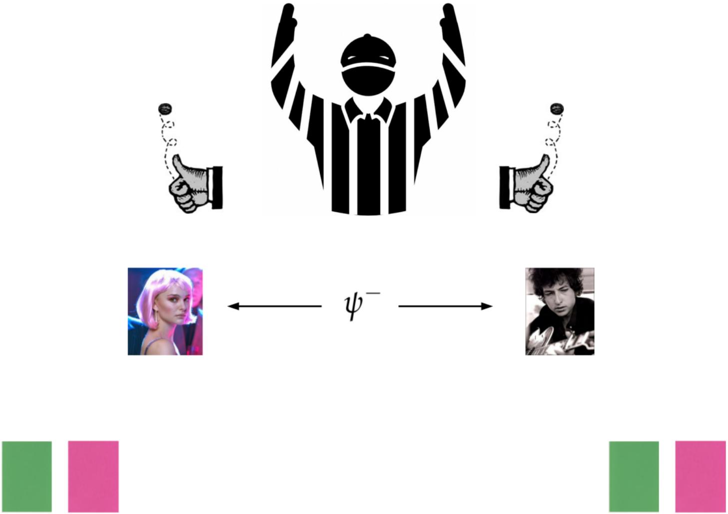
- How to prove or disprove such a thing?
- Local Hidden Variable models (LHV)

Quantum Scenarios



Local hidden variable model

$$W_\eta := \eta|\phi^+\rangle\langle\phi^+| + (1-\eta)\frac{I}{4}$$

$$|\phi^+\rangle := \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

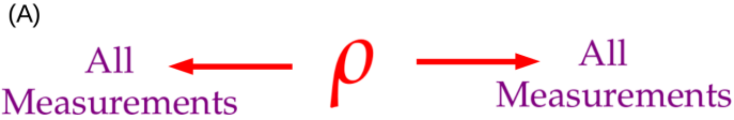
Local hidden variable model

$$W_\eta := \eta|\phi^+\rangle\langle\phi^+| + (1-\eta)\frac{I}{4}$$

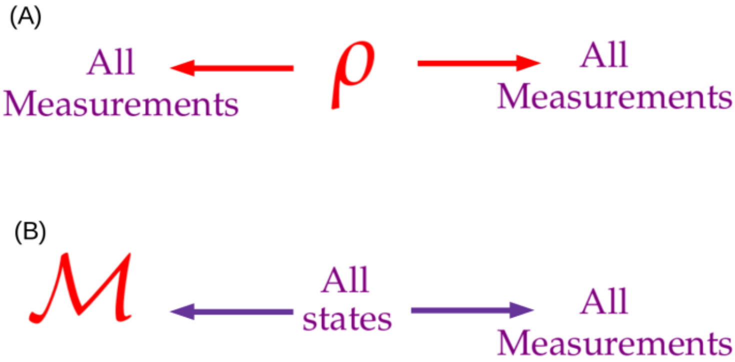
For $\eta \leq 1/2$, there exists a Local Hidden Variable model

R. Werner, PRA (1989)

Local hidden variable model



Local hidden variable model



Bell nonlocality

$$p(ab|xy) = \mathrm{tr}(\rho^{\eta}_{AB}\, A_{a|x} \otimes B_{b|y})$$

$$\rho^{\eta}_{AB} := \eta\rho_{AB} + (1-\eta)\frac{I}{d} \otimes \rho_B$$

$$\rho_B := \mathrm{tr}_A(\rho_{AB})$$

Bell nonlocality

$$\mathrm{tr}(\rho_{AB}^{\eta}\, A_{a|x} \otimes B_{b|y}) = \mathrm{tr}(\rho_{AB}\, A_{a|x}^{\eta} \otimes B_{b|y})$$

$$A_{a|x}^{\eta} := \eta A_{a|x} + (1-\eta)\frac{I}{d}\,\mathrm{tr}(A_{a|x})$$

Local hidden variable model

LHV model for the class:

$$W_{\eta,\theta} := \eta|\phi^{\theta}\rangle\langle\phi^{\theta}| + (1-\eta)\frac{I}{2} \otimes \rho_{\theta}$$

$$|\phi^{\theta}\rangle := \sin(\theta)|00\rangle + \cos(\theta)|11\rangle, \quad \rho_{\theta} := \mathrm{tr}_A(|\phi^{\theta}\rangle\langle\phi^{\theta}|$$

in a range where there are noisy incompatible measurements

Local hidden variable model

$$W_{\eta,\theta} := \eta|\phi^{\theta}\rangle\langle\phi^{\theta}| + (1-\eta)\frac{I}{2} \otimes \rho_{\theta}$$

$$\cos^2(\theta) \geq \frac{2\eta - 1}{(2-\eta)\eta^3} \implies \text{LHV model}$$

J. Bowles, F. Hirsch, M.T. Quintino, N. Brunner, PRL (2016)

Local hidden variable model

$$W_{\eta,\theta} := \eta|\phi^\theta\rangle\langle\phi^\theta| + (1-\eta)\frac{I}{2} \otimes \rho_\theta$$

$$\cos^2(\theta) \geq \frac{2\eta - 1}{(2-\eta)\eta^3} \implies \text{LHV model}$$

For $\eta > 1/2$, $\{A_{a|x}\} \implies$ not JM
But for $\theta = \pi/4$ the model returns $\eta = 1/2\ldots$

Local hidden variable model

For the maximally entangled state, we have Grothendieck!

Local hidden variable model

Journal of Soviet Mathematics,
vol. 36, number 4, pp. 557–570.
February, 1987

QUANTUM ANALOGUES OF THE BELL INEQUALITIES. THE CASE OF TWO SPATIALLY SEPARATED DOMAINS

B. S. Tsirel'son

UDC 519.2

significantly simpler manner in terms of vectors in a Euclidean space. Then it becomes clear that the constant $K_{\cdot}$ is nothing else but Grothendieck's well known constant K_G, investigated by mathematicians from 1956 up to now!

Regarding the problem of the Grothendieck constant, we refer the reader to [11]. It is proved there that

$$K_G \lesssim \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1.782.$$

Local hidden variable model

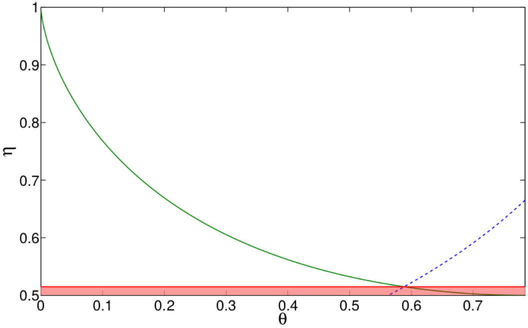
$$W_\eta := \eta|\phi^+\rangle\langle\phi^+| + (1-\eta)\frac{I}{4}$$

LHV for $\eta \leq \frac{1}{K_G(3)}$, and $K_G(3) < 2$

B. Tsirelson, J. Soviet Math. (1987)
Better local hidden variable models for two-qubit Werner states and an upper bound on the Grothendieck constant $K_G(3)$ F Hirsch, MT Quintino, T Vértesi, M Navascués, N Brunner

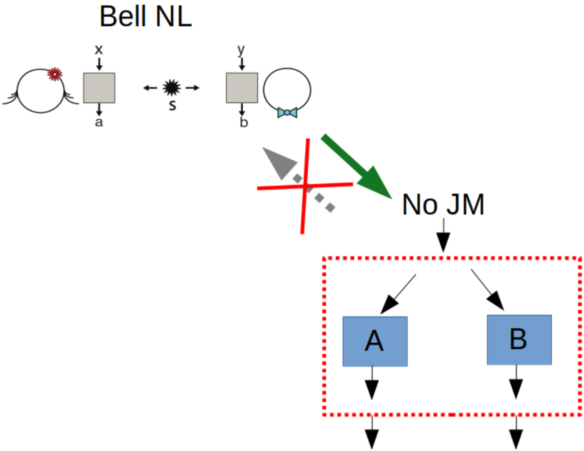
Bell NL and JM

LHV Model+Grothendieck+LHV extention based on PPT:



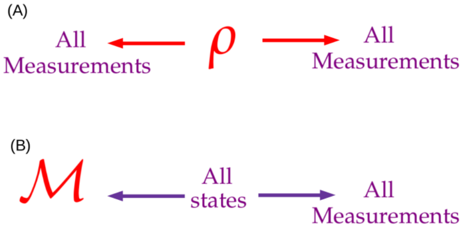
M.T. Quintino, J. Bowles, F. Hirsch, N. Brunner, PRA (2016)

Bell NL and JM



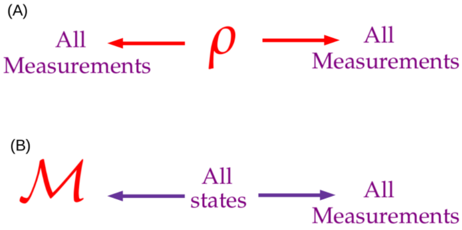
Bell NL and JM

- "Hidden" projective measurement assumption...



Bell NL and JM

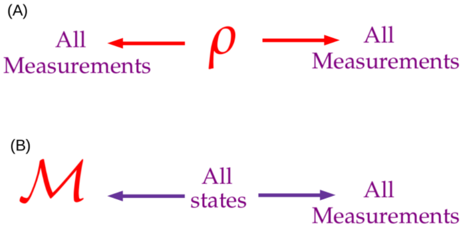
- “Hidden” projective measurement assumption. . .



- Creating LHV models requires creativity. . .
  and we are lacking it now

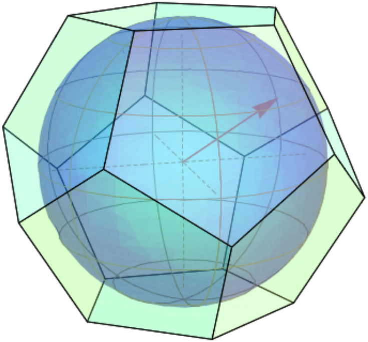
Bell NL and JM

- "Hidden" projective measurement assumption. . .



- Creating LHV models requires creativity. . .
  and we are lacking it now
- How about using the computer to find LHV models for us?

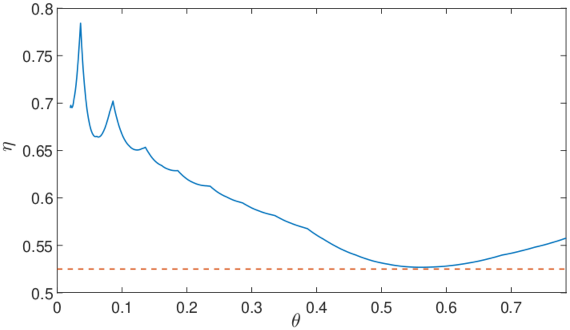
Algorithmic constructing LHV models



M.T. Quintino, F. Hirsch, T. Vertesi, M. Pusey, N. Brunner, PRL (2016)
D. Cavalcanti, L. Guerini, R. Rabelo, P. Skrzypczyk, PRL (2016)
J Bavaresco, MT Quintino, L Guerini, TO Maciel, D Cavalcanti, MT Cunha, PRA, 2017

LHV models for POVMs

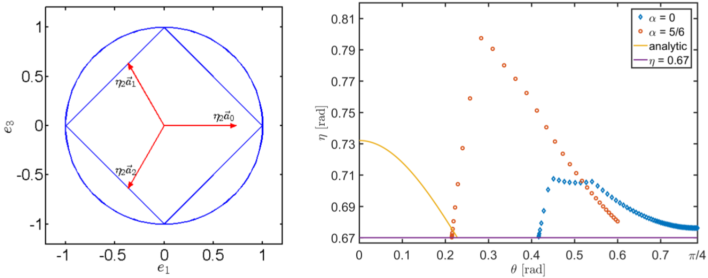
$$W_{\eta,\theta} := \eta|\phi^\theta\rangle\langle\phi^\theta| + (1-\eta)\frac{I}{2} \otimes \rho_\theta$$



F. Hirsch, M.T. Quintino, N. Brunner, PRA (2018)

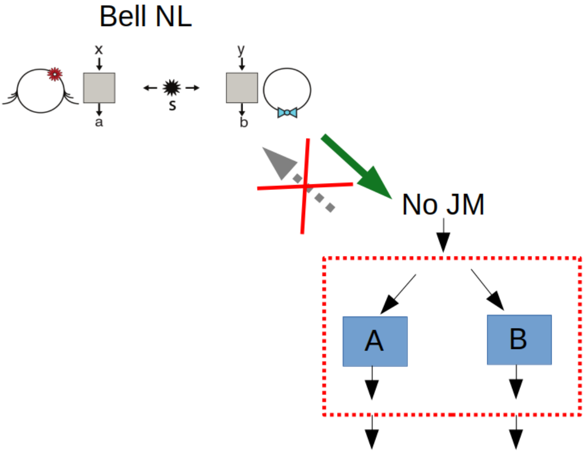
Using similar methods:

Three planar measurements:



E. Bene, T. Vertesi, NJP (2018)

Bell NL and JM



Bell NL and JM

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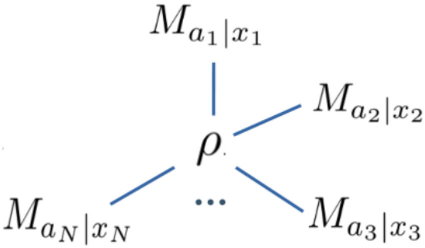
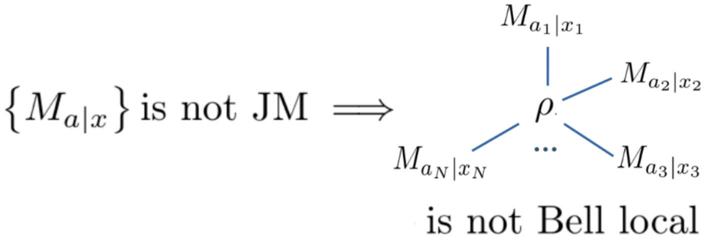
- Imagine an N-partite Bell scenario where All parties perform the same measurement



Multipartite Bell NL and JM



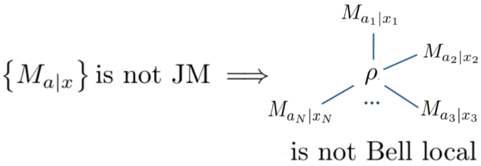
All incompatible measurements on qubits lead to multiparticle Bell nonlocality
M. Plávala, O. Gühne, M.T. Quintino arXiv (2024)

Multipartite Bell NL and JM

If $d = 2$ and $\{M_{a|x}\}$ is not JM, there exists a number of parties N and a state ρ such that

$$p(a_1 \ldots a_N | x_1 \ldots x_N) = \text{tr}(\rho\, M_{a_1|x_1} \otimes \ldots \otimes M_{a_N|x_N})$$

is Bell NL



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Bell NL and JM

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$$p(abc|xyz) = \sum_{\lambda} p(\lambda) p_A(a|x\lambda) p_{BC}(bc|yz\lambda)$$

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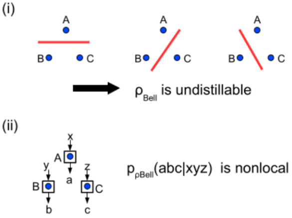

Anonymous quantum nonlocality
YC Liang, FJ Curchod, J Bowles, N Gisin
PRL 2024

Applications

New operational quantifier:

Definition 4. *Let* $\mathsf{M} = \{M_{a|x}\}$ *be a set of POVMs acting in* $\mathbb{C}_d$. *Its noisy k-partite Bell joint measurability* $\eta_k^{\mathrm{Bell}}(\mathsf{M}) \in [0,1]$ *is defined as*

$$\begin{aligned} \eta_k^{\mathrm{Bell}}(\mathsf{M}) &:= \max \eta, \\ \textit{such that} &: \forall \rho \in \mathcal{D}(\mathbb{C}_d^{\otimes k}) \\ &\mathrm{Tr}[\rho(M^{(\eta)}_{a_1|x_1} \otimes \ldots \otimes M^{(\eta)}_{a_k|x_k})] \textit{ is Bell local.} \end{aligned} \quad (8)$$

Applications

New operational quantifier:

Corollary 6. *If $\eta > \frac{1}{2}$, there exists a $N \in \mathbb{N}$, and a N qubit state $\rho \in \mathcal{D}(\mathbb{C}_2^{\otimes N})$ such that the state $D_\eta^{\otimes N}(\rho)$ is entangled and violates a Bell inequality.*

Proof methods

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  Annales Henri Poincaré 20, 2295 (2019)

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- Recognise that, measurements lead to Bell local correlations of qubit states iff they're generalsing annihilating channels.

Open questions

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- How does measurement locality relate to other areas of quantum info?

Thank you!

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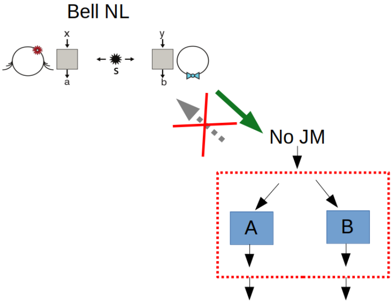
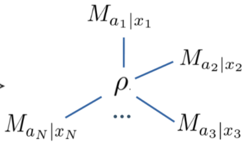


$$\{M_{a|x}\} \text{ is not JM} \implies$$



is not Bell local