All incompatible measurements on qubits lead to multiparticle Bell nonlocality

Martin Plávala, Otfried Gühne, Marco Túlio Quintino

Sorbonne Université, CNRS, LIP6

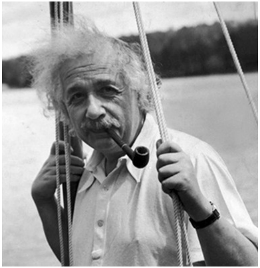
June 25, 2024

Measurement incompatibility

$$\Delta x \, \Delta p \geq \hbar/2$$

Quantum entanglement

$$\rho_{AB} \neq \int \pi(\lambda) \rho_A^\lambda \otimes \rho_B^\lambda \, \mathrm{d}\lambda$$




State+Measurement

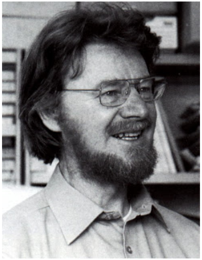
$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i)$$

State+Measurement

$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i), \quad |\langle i|\psi\rangle|^2$$

Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$



Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

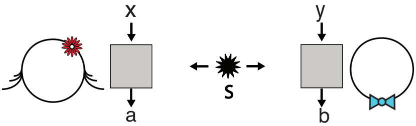
Bell NL $\implies$ Entanglement + Measurement incompatibility

Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

$$\text{Bell NL} \implies \text{Entanglement} + \text{Measurement incompatibility}$$
$$\text{Bell NL} \stackrel{?}{\impliedby} \text{Entanglement} + \text{Measurement incompatibility}$$

Bell Nonlocality



The Correlation/Anticorrelation Game

The Correlation/Anticorrelation Game

The Correlation/Anticorrelation Game

The Correlation/Anticorrelation Game

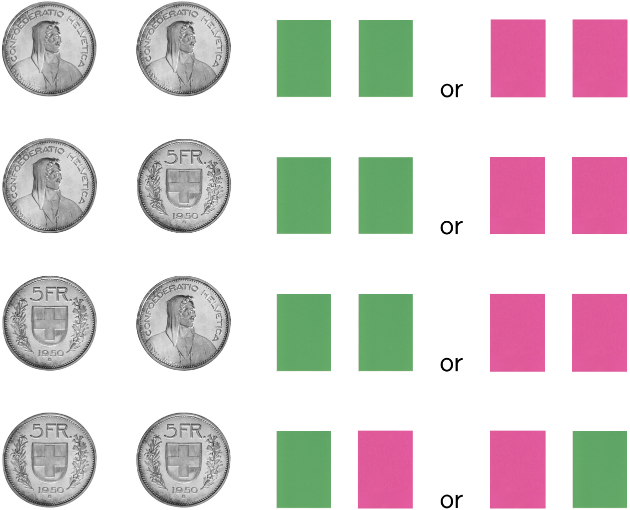
The Correlation/Anticorrelation Game

The Correlation/Anticorrelation Game

The Correlation/Anticorrelation Game

The Correlation/Anticorrelation Game

Winning Conditions



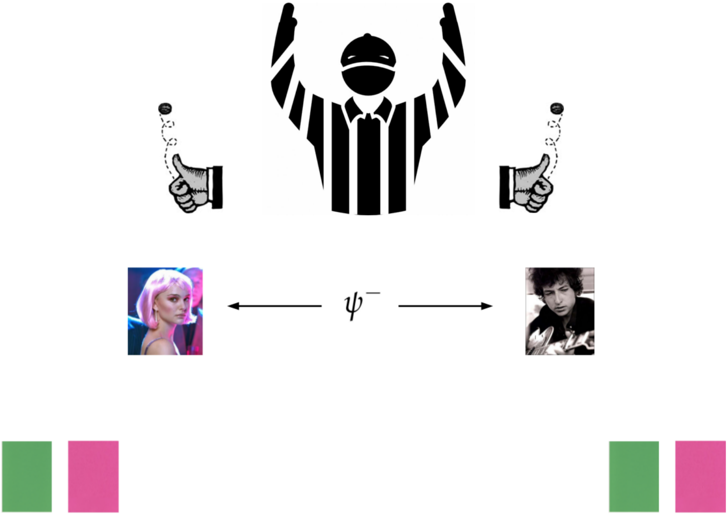
Best Strategy

Can Alice and Bob always win?

Best Strategy

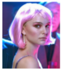
Best (classical) strategy wins with probability $\frac{3}{4}$

Quantum Strategy



Quantum Strategy

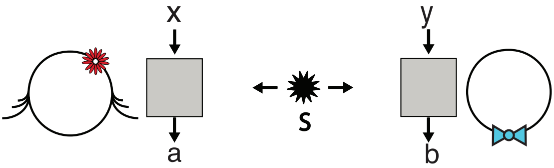
$$p_q = \frac{2+\sqrt{2}}{4} \approx 0.8535$$



$\longleftarrow \psi^- \longrightarrow$

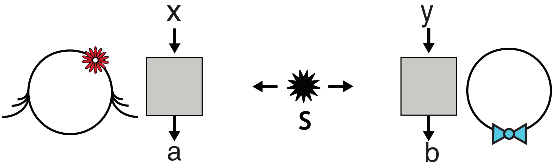
Bell nonlocality

$$p_{\text{win}} = \sum_{abxy} \pi(x,y) V(ab|xy) p(ab|xy)$$



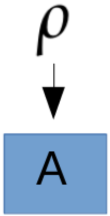
Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

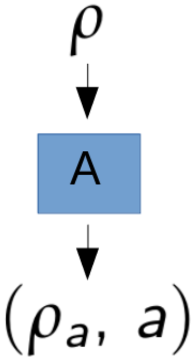


Quantum measurement

Quantum measurement



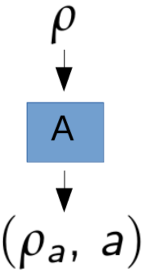
Quantum measurement



Quantum measurement: POVM

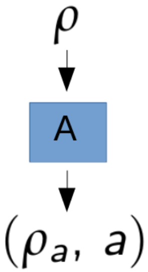
$$p(a|\rho, A) = \mathrm{tr}(\rho A_a)$$

$$A = \{A_a\}, \quad A_a \geq 0, \quad \sum_a A_a = I$$



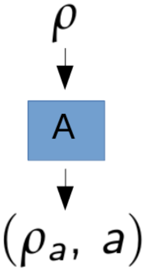
Quantum measurement

$$p(a|\rho, A) = \mathrm{tr}(\rho A_a), \quad \rho_a = A_a \text{ (if projective)}$$

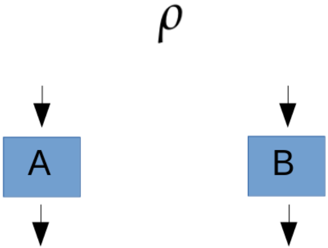


Quantum measurement

$$p(a|\rho, A) = \mathrm{tr}(\rho A_a), \quad \rho_a = \frac{\sqrt{A_a}\rho\sqrt{A_a}}{\mathrm{tr}(\rho A_a)} \text{ (Lüders)}$$

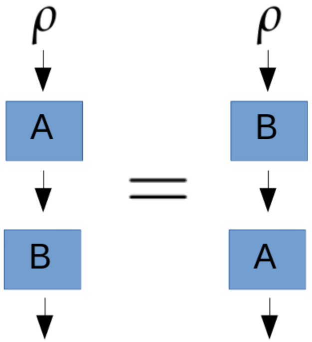


Measurement compatibility



Measurement compatibility

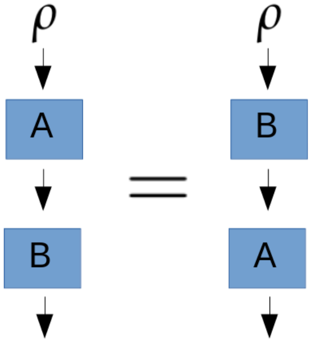
When the measurements commute:



Measurement compatibility

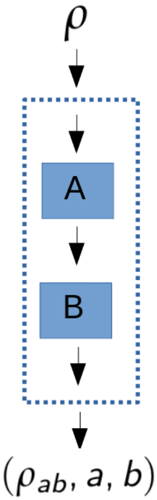
When the measurements commute:

$$[A_a, B_b] := A_a B_b - B_b A_a = 0$$



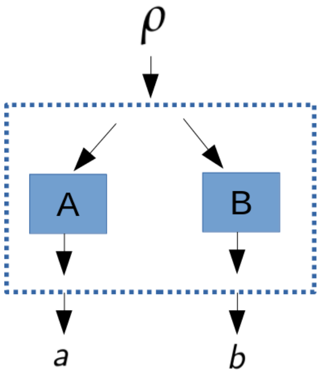
Measurement compatibility

Commutation $\implies$ measurement compatibility



Measurement compatibility

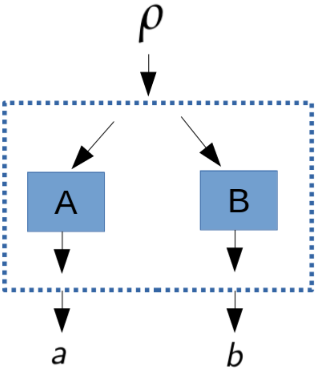
Joint measurability



Joint Measurability

$\{A_a\}$ and $\{B_b\}$ are JM if there exists a measurement $\{M_{ab}\}$ s. t.:

$$\sum_a M_{ab} = B_b$$
$$\sum_b M_{ab} = A_a$$

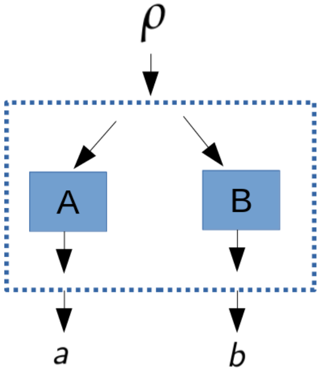


Joint Measurability

$\{A_a\}$ and $\{B_b\}$ are JM if there exists a single measurement $\{M_{ab}\}$ s. t.:

$$\sum_a M_{ab} = B_b$$
$$\sum_b M_{ab} = A_a$$

$$M_{ab} \geq 0, \quad \sum_{ab} M_{ab} = I$$



Joint Measurability

The set of measurements $A_{a|x}$ is JM if there exists a single measurement $\{M_\lambda\}$ and a classical post-processing $p(a|x,\lambda)$ s. t.:

$$A_{a|x} = \sum_\lambda p(a|x,\lambda) M_\lambda$$

Pauli Measurements

$$\sigma_Z : \{|0\rangle\langle 0|,\ |1\rangle\langle 1|\} \quad \sigma_X : \{|+\rangle\langle +|,\ |-\rangle\langle -|\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2} \, ; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2}\,; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{X,\eta} : \left\{ \eta|+\rangle\langle +| + (1-\eta)\frac{I}{2}\,; \qquad \eta|-\rangle\langle -| + (1-\eta)\frac{I}{2} \right\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2}\ ; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{X,\eta} : \left\{ \eta|+\rangle\langle +| + (1-\eta)\frac{I}{2}\ ; \qquad \eta|-\rangle\langle -| + (1-\eta)\frac{I}{2} \right\}$$

$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Joint Measurability}$$

Hollow Triangle

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2} \; ; \quad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{X,\eta} : \left\{ \eta|+\rangle\langle +| + (1-\eta)\frac{I}{2} \; ; \quad \eta|-\rangle\langle -| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{Y,\eta} : \left\{ \eta|Y+\rangle\langle Y+| + (1-\eta)\frac{I}{2} \; ; \quad \eta|Y-\rangle\langle Y-| + (1-\eta)\frac{I}{2} \right\}$$

$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Pairwise Measurability}$$

Hollow Triangle

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2}\,; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

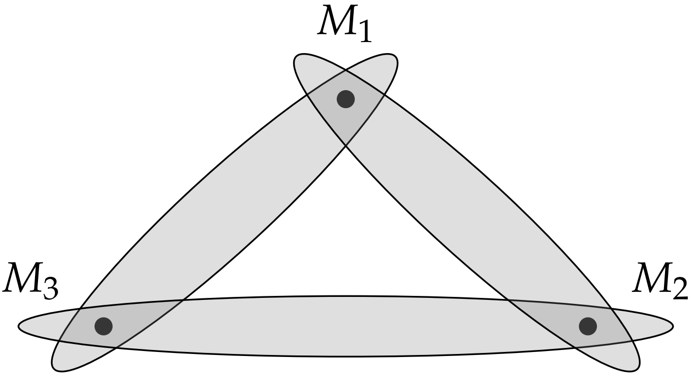
$$\sigma_{X,\eta} : \left\{ \eta|+\rangle\langle +| + (1-\eta)\frac{I}{2}\,; \qquad \eta|-\rangle\langle -| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{Y,\eta} : \left\{ \eta|Y+\rangle\langle Y+| + (1-\eta)\frac{I}{2}\,; \qquad \eta|Y-\rangle\langle Y-| + (1-\eta)\frac{I}{2} \right\}$$

$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Pairwise Measurability}$$

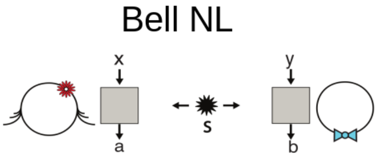
$$\eta \leq \frac{1}{\sqrt{3}} \implies \text{Triplewise Measurability}$$

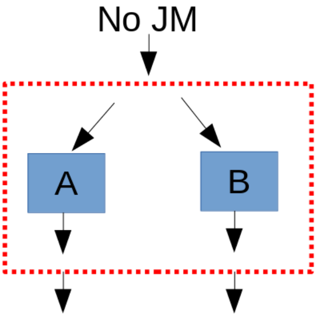
Hollow Triangle



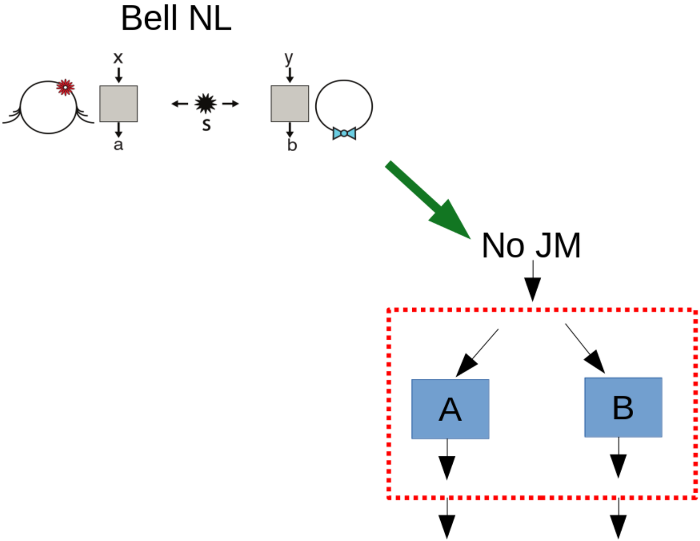
T. Heinosaari, D. Reitzner, P. Stano: Foundations of Physics (2008)

Bell NL and no JM

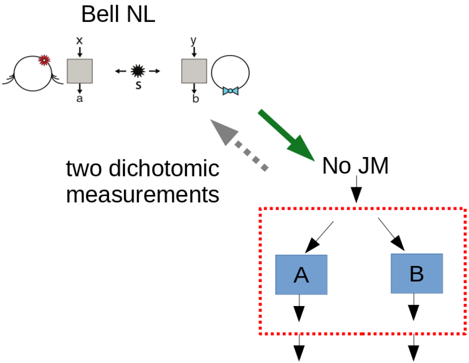




Bell NL and no JM



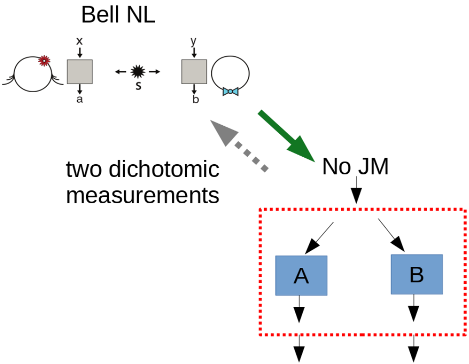
Bell NL and JM



M. Wolf, D. Perez-Garcia, C. Fernandez, PRL (2009)

Bell NL and JM

$\{A_{a|x}\}_{a,x=1}^{2}$ not JM $\implies$ $\exists \rho_{AB}$ and $\{B_{b|y}\}$ such that:
$p(ab|xy) = \mathrm{tr}(\rho_{AB}\, A_{a|x} \otimes B_{b|y})$ is Bell NL



M. Wolf, D. Perez-Garcia, C. Fernandez, PRL (2009)

Bell nonlocality

- Are all incompatible measurements useful for Bell NL?

Bell nonlocality

- Are all incompatible measurements useful for Bell NL?
- All incompatible measurements useful for EPR steering!
  Joint measurability, EPR steering, and Bell nonlocality
  MT. Quintino, T. Vértesi, N. Brunner, PRL (2015)

  Joint measurability of generalized measurements implies classicality
  R Uola, T Moroder, O Gühne, PRL (2015)

Bell nonlocality

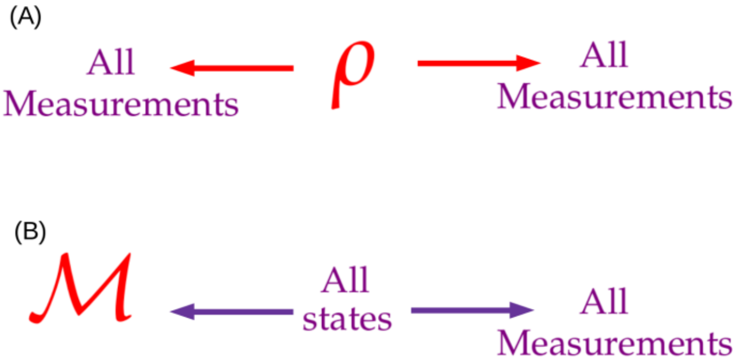
- Are all incompatible measurements useful for Bell NL?
- All incompatible measurements useful for EPR steering!
  Joint measurability, EPR steering, and Bell nonlocality
  MT. Quintino, T. Vértesi, N. Brunner, PRL (2015)

  Joint measurability of generalized measurements implies classicality
  R Uola, T Moroder, O Gühne, PRL (2015)
- But, some sets of incompatible measurements are "useless" for Bell NL...

Local hidden variable model

(A)

Local hidden variable model



Bell local measurements

- LHV model for a set of all noisy qubit projective measurements (dichotomic assumption)
  Incompatible quantum measurements admitting a local hidden variable model
  M.T. Quintino, J. Bowles, F. Hirsch, N. Brunner, PRA, 2016

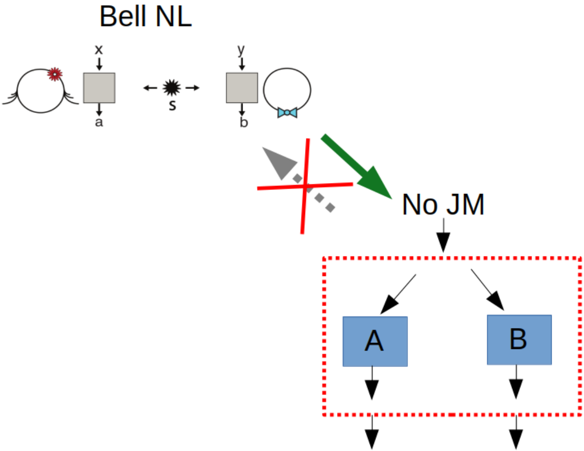
Bell local measurements

- LHV model for a set of all noisy qubit projective measurements (dichotomic assumption)
  Incompatible quantum measurements admitting a local hidden variable model
  M.T. Quintino, J. Bowles, F. Hirsch, N. Brunner, PRA, 2016
- LHV model for a set of all noisy qubit projective measurements
  Quantum measurement incompatibility does not imply Bell nonlocality
  F. Hirsch, M. T. Quintino, N. Brunner PRA, 2018

Bell local measurements

- LHV model for a set of all noisy qubit projective measurements (dichotomic assumption)
  Incompatible quantum measurements admitting a local hidden variable model
  M.T. Quintino, J. Bowles, F. Hirsch, N. Brunner, PRA, 2016
- LHV model for a set of all noisy qubit projective measurements
  Quantum measurement incompatibility does not imply Bell nonlocality
  F. Hirsch, M. T. Quintino, N. Brunner PRA, 2018
- LHV model for a qubit trine
  Measurement incompatibility does not give rise to Bell violation in general
  E. Bene, T. Vertesi, NJP (2018)

Bell NL and JM



Bell NL and JM

- $\{M_{a|x}\}$ is not JM, but useless for Bell NL.

Bell NL and JM

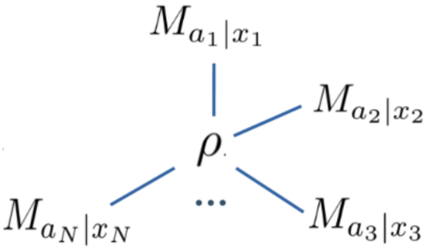
- $\{M_{a|x}\}$ is not JM, but useless for Bell NL.
- $\{M_{a|x}\}$ is not JM, but useless for **bipartite** Bell NL.

Bell NL and JM

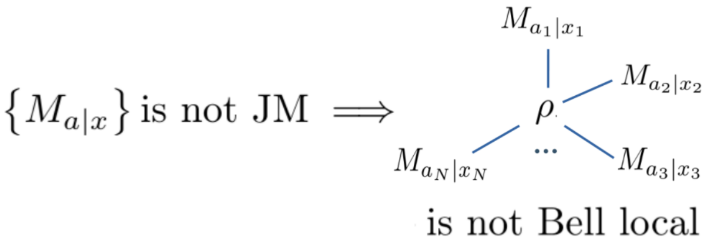
- $\{M_{a|x}\}$ is not JM, but useless for Bell NL.
- $\{M_{a|x}\}$ is not JM, but useless for **bipartite** Bell NL.
- How about multipartite scenarios?

Bell NL and JM

- $\{M_{a|x}\}$ is not JM, but useless for Bell NL.
- $\{M_{a|x}\}$ is not JM, but useless for **bipartite** Bell NL.
- How about multipartite scenarios?
- How about an N-partite Bell scenario where All parties perform the same measurement



Multipartite Bell NL and JM



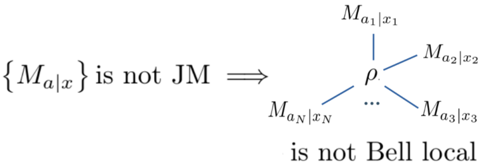
All incompatible measurements on qubits lead to multiparticle Bell nonlocality
M. Plávala, O. Gühne, M.T. Quintino arXiv (2024)

Multipartite Bell NL and JM

If $d = 2$ and $\{M_{a|x}\}$ is not JM, there exists a number of parties N and a state ρ such that

$$p(a_1 \dots a_N | x_1 \dots x_N) = \mathrm{tr}(\rho \, M_{a_1|x_1} \otimes \dots \otimes M_{a_N|x_N})$$

is Bell NL



All incompatible measurements on qubits lead to multiparticle Bell nonlocality
M. Plávala, O. Gühne, M.T. Quintino arXiv (2024)

Applications

- New JM quantifier for qubits:
  How many parties do you need to display Bell NL ?

Applications

- New JM quantifier for qubits:
  How many parties do you need to display Bell NL ?
- Corollary:
  Let $D_\eta(\rho) := \eta\rho + (1-\eta)\frac{\mathbb{1}}{d}$ be the depolarising map. If $\eta > \frac{1}{2}$, there exists a N-partite quantum state ρ_N such that

$$D_\eta^{\otimes N}(\rho_N) = D_\eta \otimes D_\eta \dots D_\eta(\rho_N)$$

  is Bell NL.

Superactivation of Bell JM

- Let $\{A_{a|x}\}$ be a set of incompatible qubit measurements with LHV for bipartite Bell NL

Superactivation of Bell JM

- Let $\{A_{a|x}\}$ be a set of incompatible qubit measurements with LHV for bipartite Bell NL
- We can "activate" its nonlocality in a multipartite scenario

Superactivation of Bell JM

- Let $\{A_{a|x}\}$ be a set of incompatible qubit measurements with LHV for bipartite Bell NL
- We can "activate" its nonlocality in a multipartite scenario
- **Not** genuine multipartite Bell NL

$$p(abc|xyz) = \sum_{\lambda} p(\lambda) p_A(a|x\lambda) p_{BC}(bc|yz\lambda)$$

Superactivation of Bell JM

- Let $\{A_{a|x}\}$ be a set of incompatible qubit measurements with LHV for bipartite Bell NL
- We can "activate" its nonlocality in a multipartite scenario
- **Not** genuine multipartite Bell NL

$$p(abc|xyz) = \sum_{\lambda} p(\lambda) p_A(a|x\lambda) p_{BC}(bc|yz\lambda)$$

- But definitely Bell NL

$$p(abc|xyz) \neq \sum_{\lambda} p(\lambda) p_A(a|x\lambda) p_B(b|y\lambda) p_C(c|z\lambda)$$

Superactivation of Bell JM

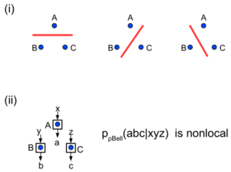
- Let $\{A_{a|x}\}$ be a set of incompatible qubit measurements with LHV for bipartite Bell NL
- We can "activate" its nonlocality in a multipartite scenario
- **Not** genuine multipartite Bell NL

$$p(abc|xyz) = \sum_{\lambda} p(\lambda) p_A(a|x\lambda) p_{BC}(bc|yz\lambda)$$

- But definitely Bell NL

$$p(abc|xyz) \neq \sum_{\lambda} p(\lambda) p_A(a|x\lambda) p_B(b|y\lambda) p_C(c|z\lambda)$$

- Anonymous NL:



T.Vertesi, N. Brunner, PRL (2012)
YC Liang, FJ Curchod, J Bowles, N Gisin, PRL (2014)

Proof methods

- View Bell locality as separability in Generalised Probabilistic Theories (GPT)

Proof methods

- View Bell locality as separability in Generalised Probabilistic Theories (GPT)
- View quantum measurements as maps transforming states into probabilities $\mathcal{M}(\rho) = \{\mathrm{tr}(\rho M_{a|x})\}_{ax}$

Proof methods

- View Bell locality as separability in Generalised Probabilistic Theories (GPT)
- View quantum measurements as maps transforming states into probabilities $\mathcal{M}(\rho) = \{\mathrm{tr}(\rho M_{a|x})\}_{ax}$
- $\mathcal{M}$ is "entanglement breaking" iff

$$\mathcal{M}(\rho) = \left\{ \sum_{\lambda} \mathrm{tr}(E_{\lambda}\rho) p(a|x, \lambda) \right\}_{ax}$$

Proof methods

- View Bell locality as separability in Generalised Probabilistic Theories (GPT)
- View quantum measurements as maps transforming states into probabilities $\mathcal{M}(\rho) = \{\mathrm{tr}(\rho M_{a|x})\}_{ax}$
- $\mathcal{M}$ is "entanglement breaking" iff

$$\mathcal{M}(\rho) = \left\{ \sum_{\lambda} \mathrm{tr}(E_{\lambda}\rho) p(a|x,\lambda) \right\}_{ax}$$

- $\mathcal{M}$ is "entanglement annihilating" iff $\mathcal{M}^{\otimes N}(\rho)$ is separable $\forall N \in \mathbb{N}$.

Proof methods

- View Bell locality as separability in Generalised Probabilistic Theories (GPT)
- View quantum measurements as maps transforming states into probabilities $\mathcal{M}(\rho) = \{\mathrm{tr}(\rho M_{a|x})\}_{ax}$
- $\mathcal{M}$ is "entanglement breaking" iff

$$\mathcal{M}(\rho) = \left\{ \sum_{\lambda} \mathrm{tr}(E_\lambda \rho) p(a|x, \lambda) \right\}_{ax}$$

- $\mathcal{M}$ is "entanglement annihilating" iff $\mathcal{M}^{\otimes N}(\rho)$ is separable $\forall N \in \mathbb{N}$.
- Generalise a new result on entanglement breaking channels and entanglement annihilating channels.

When do composed maps become entanglement breaking?
M. Christandl, A. Müller-Hermes, and M. M. Wolf
Annales Henri Poincaré 20, 2295 (2019)

Proof methods

- View Bell locality as separability in Generalised Probabilistic Theories (GPT)
- View quantum measurements as maps transforming states into probabilities $\mathcal{M}(\rho) = \{\mathrm{tr}(\rho M_{a|x})\}_{ax}$
- $\mathcal{M}$ is "entanglement breaking" iff

$$\mathcal{M}(\rho) = \left\{ \sum_{\lambda} \mathrm{tr}(E_{\lambda}\rho) p(a|x,\lambda) \right\}_{ax}$$

- $\mathcal{M}$ is "entanglement annihilating" iff $\mathcal{M}^{\otimes N}(\rho)$ is separable $\forall N \in \mathbb{N}$.
- Generalise a new result on entanglement breaking channels and entanglement annihilating channels.
  When do composed maps become entanglement breaking?
  M. Christandl, A. Müller-Hermes, and M. M. Wolf
  Annales Henri Poincaré 20, 2295 (2019)
- Recognise that, measurements lead to Bell local correlations of qubit states iff they're generalising annihilating channels.

Open questions

- JM and multipartite Bell NL for $d > 2$?

Open questions

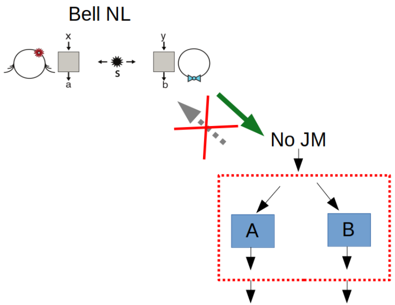
- JM and multipartite Bell NL for $d > 2$?
- Direct/intuitive LHV model for incompatible measurements?

Open questions

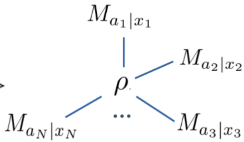
- JM and multipartite Bell NL for $d > 2$?
- Direct/intuitive LHV model for incompatible measurements?
- Simple/useful criteria for measurement Bell NL?

Thank you!

Thank you!



$\{M_{a|x}\}$ is not JM $\Longrightarrow$



is not Bell local