All incompatible measurements on qubits lead to multiparticle Bell nonlocality

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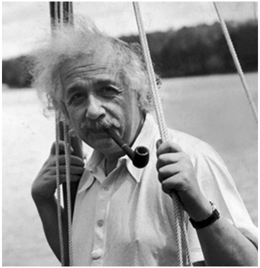
Measurement incompatibility

$$\Delta x \, \Delta p \geq \hbar/2$$

Quantum entanglement

$$\rho_{AB} \neq \int \pi(\lambda)\rho_A^\lambda \otimes \rho_B^\lambda \, \mathrm{d}\lambda$$



State+Measurement

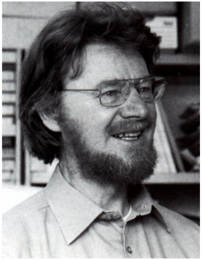
$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i)$$

State+Measurement

$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i), \quad |\langle i|\psi\rangle|^2$$

Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$



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Bell NL $\implies$ Entanglement + Measurement incompatibility

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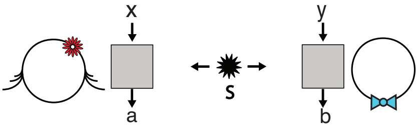
$$\text{Bell NL} \implies \text{Entanglement} + \text{Measurement incompatibility}$$
$$\text{Bell NL} \stackrel{?}{\impliedby} \text{Entanglement} + \text{Measurement incompatibility}$$

Bell Nonlocality



The Correlation/Anticorrelation Game

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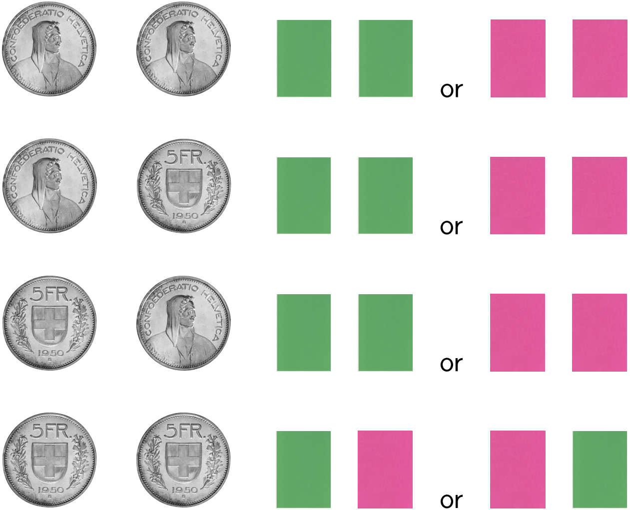
Winning Conditions



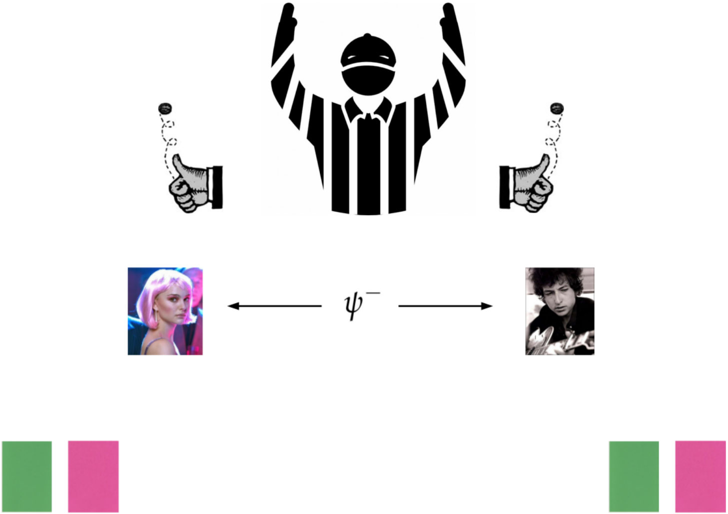
Best Strategy

Can Alice and Bob always win?

Best Strategy

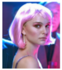
Best (classical) strategy wins with probability $\frac{3}{4}$

Quantum Strategy



Quantum Strategy

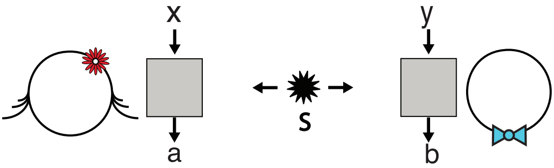
$$p_q = \frac{2+\sqrt{2}}{4} \approx 0.8535$$



$$\longleftarrow \quad \psi^- \quad \longrightarrow$$

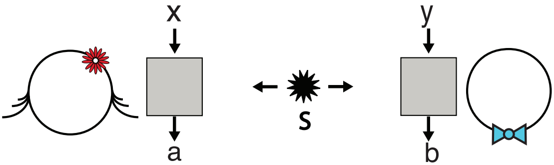
Bell nonlocality

$$p_{\text{win}} = \sum_{abxy} \pi(x,y) V(ab|xy) p(ab|xy)$$



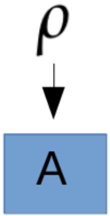
Bell nonlocality

$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

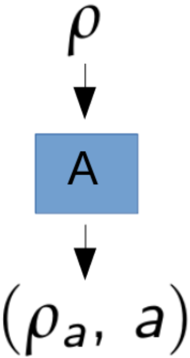


Quantum measurement

Quantum measurement



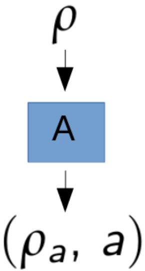
Quantum measurement



Quantum measurement: POVM

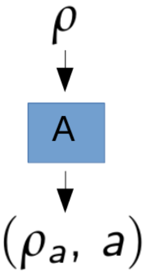
$$p(a|\rho, A) = \mathrm{tr}(\rho A_a)$$

$$A = \{A_a\}, \quad A_a \geq 0, \quad \sum_a A_a = I$$



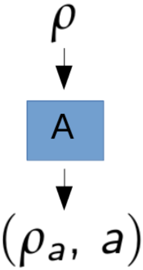
Quantum measurement

$$p(a|\rho, A) = \mathrm{tr}(\rho A_a), \quad \rho_a = A_a \text{ (if projective)}$$

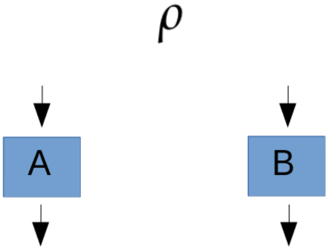


Quantum measurement

$$p(a|\rho, A) = \mathrm{tr}(\rho A_a), \quad \rho_a = \frac{\sqrt{A_a}\rho\sqrt{A_a}}{\mathrm{tr}(\rho A_a)} \text{ (Lüders)}$$

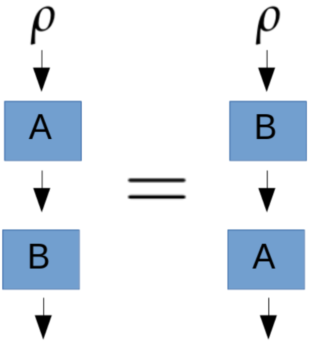


Measurement compatibility



Measurement compatibility

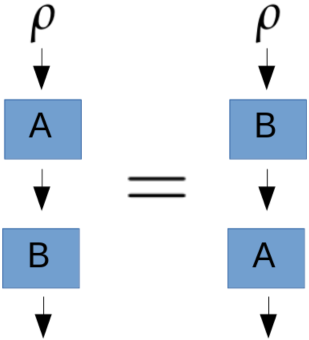
When the measurements commute:



Measurement compatibility

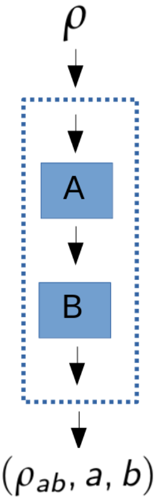
When the measurements commute:

$$[A_a, B_b] := A_a B_b - B_b A_a = 0$$



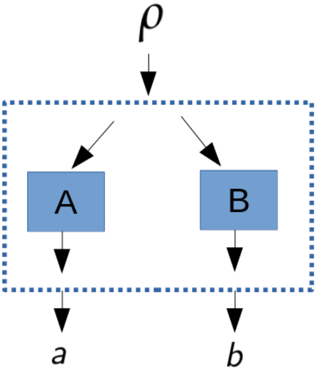
Measurement compatibility

Commutation $\implies$ measurement compatibility



Measurement compatibility

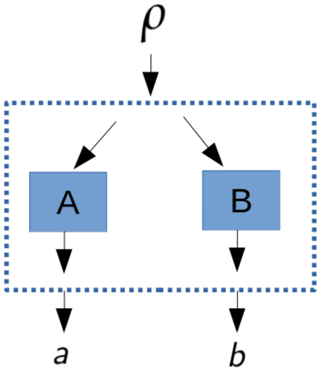
Joint measurability



Joint Measurability

$\{A_a\}$ and $\{B_b\}$ are JM if there exists a measurement $\{M_{ab}\}$ s. t.:

$$\sum_a M_{ab} = B_b$$
$$\sum_b M_{ab} = A_a$$



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$\{A_a\}$ and $\{B_b\}$ are JM if there exists a single measurement $\{M_{ab}\}$ s. t.:

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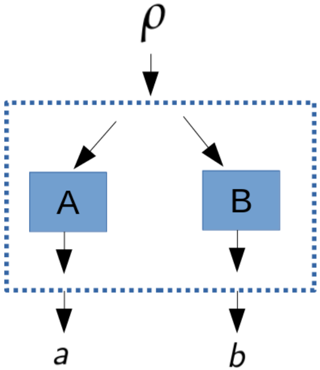
$$M_{ab} \geq 0, \quad \sum_{ab} M_{ab} = I$$



Joint Measurability

The set of measurements $A_{a|x}$ is JM if there exists a single measurement $\{M_\lambda\}$ and a classical post-processing $p(a|x,\lambda)$ s. t.:

$$A_{a|x} = \sum_\lambda p(a|x,\lambda) M_\lambda$$

Pauli Measurements

$$\sigma_Z : \{|0\rangle\langle 0|,\ |1\rangle\langle 1|\} \quad \sigma_X : \{|+\rangle\langle +|,\ |-\rangle\langle -|\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2}\,; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

Noise Pauli Measurements

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$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Joint Measurability}$$

Hollow Triangle

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$$\sigma_{Y,\eta} : \left\{ \eta|Y+\rangle\langle Y+| + (1-\eta)\frac{I}{2} \; ; \qquad \eta|Y-\rangle\langle Y-| + (1-\eta)\frac{I}{2} \right\}$$

$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Pairwise Measurability}$$

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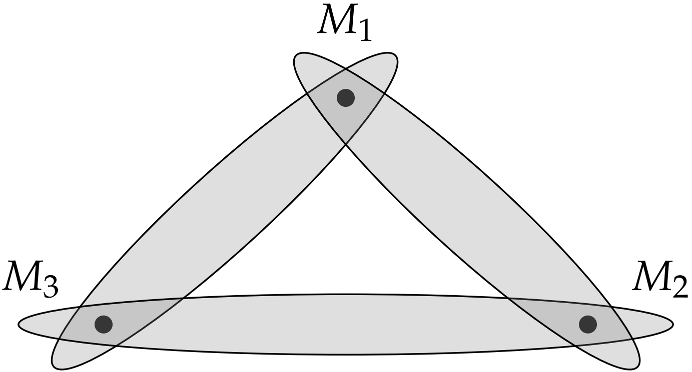
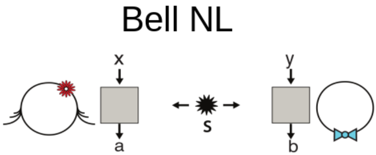
$$\eta \leq \frac{1}{\sqrt{3}} \implies \text{Triplewise Measurability}$$

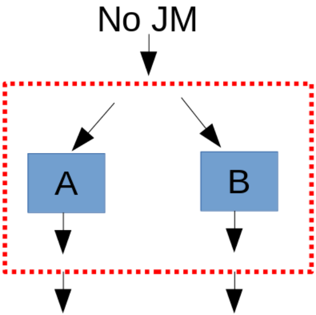
Hollow Triangle



T. Heinosaari, D. Reitzner, P. Stano: Foundations of Physics (2008)

Bell NL and no JM





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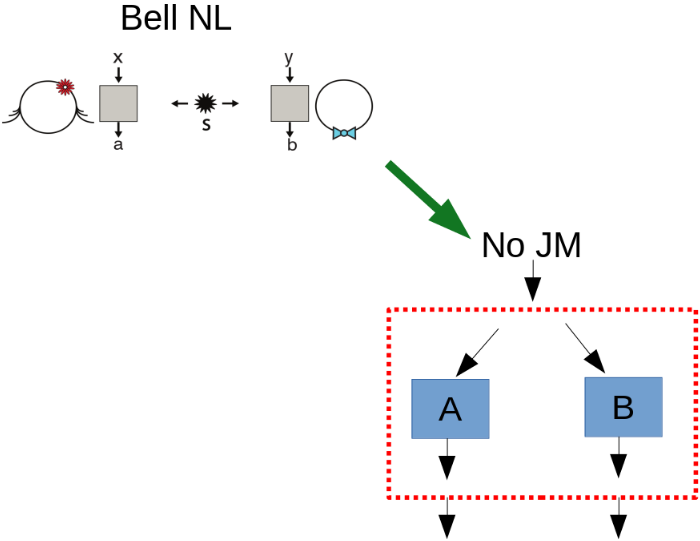
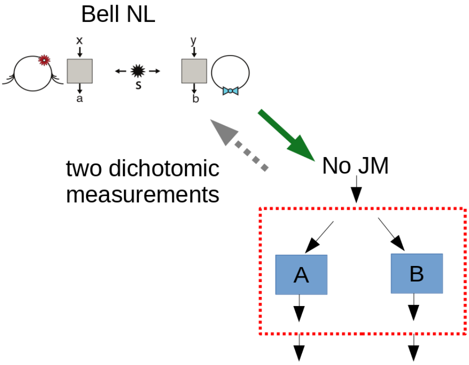


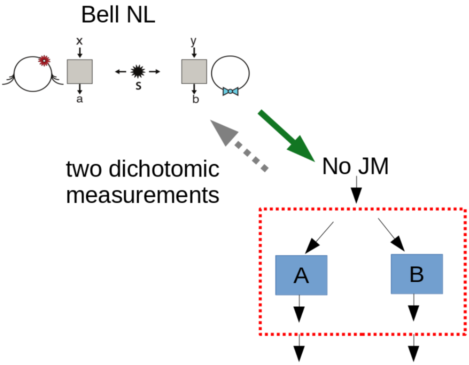
Bell NL and JM



M. Wolf, D. Perez-Garcia, C. Fernandez, PRL (2009)

Bell NL and JM

$\{A_{a|x}\}_{a,x=1}^{2}$ not JM $\implies$ $\exists \rho_{AB}$ and $\{B_{b|y}\}$ such that:
$p(ab|xy) = \mathrm{tr}(\rho_{AB}\, A_{a|x} \otimes B_{b|y})$ is Bell NL



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  Joint measurability, EPR steering, and Bell nonlocality
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  Joint measurability of generalized measurements implies classicality
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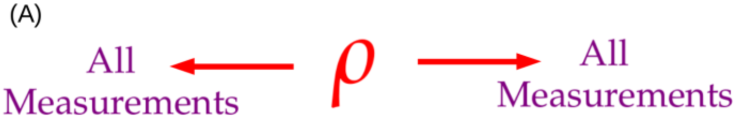
- But, some sets of incompatible measurements are "useless" for Bell NL...

Local hidden variable model



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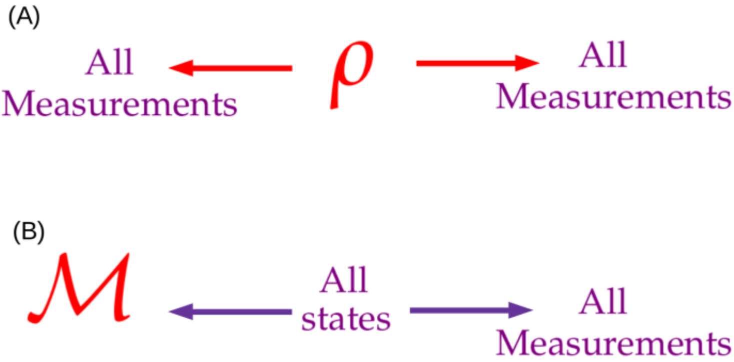

Bell local measurements

- LHV model for a set of all noisy qubit projective measurements (dichotomic assumption)
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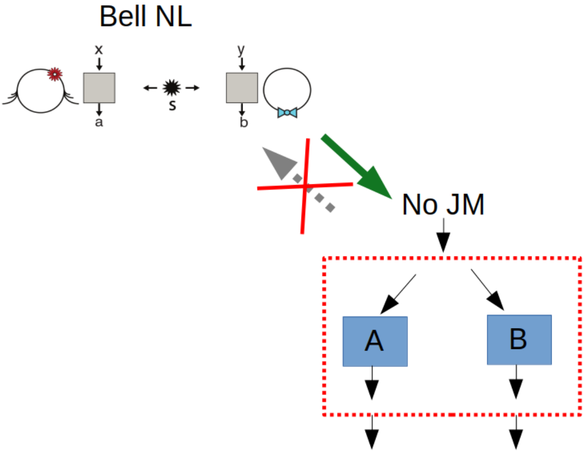
- LHV model for a qubit trine
  Measurement incompatibility does not give rise to Bell violation in general
  E. Bene, T. Vertesi, NJP (2018)

Bell NL and JM



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- How about multipartite scenarios?

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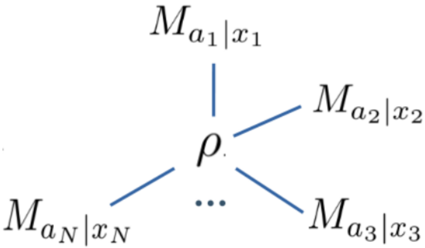
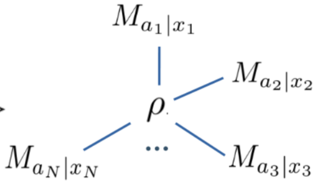
- How about multipartite scenarios?
- How about an N-partite Bell scenario where All parties perform the same measurement



Multipartite Bell NL and JM

$$\{M_{a|x}\} \text{ is not JM} \implies$$



is not Bell local

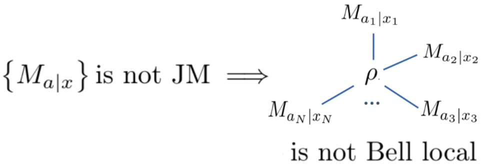
All incompatible measurements on qubits lead to multiparticle Bell nonlocality
M. Plávala, O. Gühne, M.T. Quintino arXiv (2024)

Multipartite Bell NL and JM

If $d = 2$ and $\{M_{a|x}\}$ is not JM, there exists a number of parties N and a state ρ such that

$$p(a_1 \dots a_N | x_1 \dots x_N) = \mathrm{tr}(\rho \, M_{a_1|x_1} \otimes \dots \otimes M_{a_N|x_N})$$

is Bell NL



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Applications

- New JM quantifier for qubits:
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- Corollary:
  Let $D_\eta(\rho) := \eta\rho + (1-\eta)\frac{1}{d}$ be the depolarising map. If $\eta > \frac{1}{2}$, there exists a N-partite quantum state ρ_N such that

  $$D_\eta^{\otimes N}(\rho_N) = D_\eta \otimes D_\eta \ldots D_\eta(\rho_N)$$

  is Bell NL.

Superactivation of Bell JM

- Let $\{A_{a|x}\}$ be a set of incompatible qubit measurements with LHV for bipartite Bell NL

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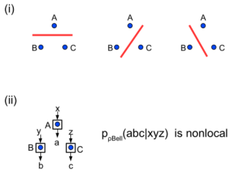
- Anonymous NL:



T.Vertesi, N. Brunner, PRL (2012)
YC Liang, FJ Curchod, J Bowles, N Gisin, PRL (2014)

Proof methods

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- Generalise a new result on entanglement breaking channels and entanglement annihilating channels.

When do composed maps become entanglement breaking?
M. Christandl, A. Müller-Hermes, and M. M. Wolf
Annales Henri Poincaré 20, 2295 (2019)

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- View quantum measurements as maps transforming states into probabilities $\mathcal{M}(\rho) = \{\mathrm{tr}(\rho M_{a|x})\}_{ax}$
- $\mathcal{M}$ is "entanglement breaking" iff

$$\mathcal{M}(\rho) = \left\{ \sum_{\lambda} \mathrm{tr}(E_{\lambda}\rho) p(a|x, \lambda) \right\}_{ax}$$

- $\mathcal{M}$ is "entanglement annihilating" iff $\mathcal{M}^{\otimes N}(\rho)$ is separable $\forall N \in \mathbb{N}$.
- Generalise a new result on entanglement breaking channels and entanglement annihilating channels.
  When do composed maps become entanglement breaking?
  M. Christandl, A. Müller-Hermes, and M. M. Wolf
  Annales Henri Poincaré 20, 2295 (2019)
- Recognise that, measurements lead to Bell local correlations of qubit states iff they're generalising annihilating channels.

Open questions

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- Direct/intuitive LHV model for incompatible measurements?
- Simple/useful criteria for measurement Bell NL?

Thank you!

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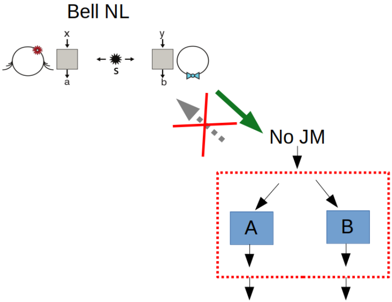
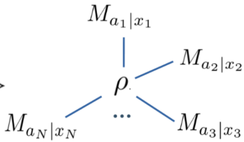


$\{M_{a|x}\}$ is not JM $\Longrightarrow$



is not Bell local