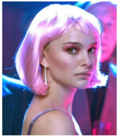
Simulating qubit correlations with classical communication

Martin Renner, Armin Tavakoli, Marco Túlio Quintino

Sorbonne Université, CNRS, LIP6

PRL 130, 120801 (2023), arXiv:2207.02244

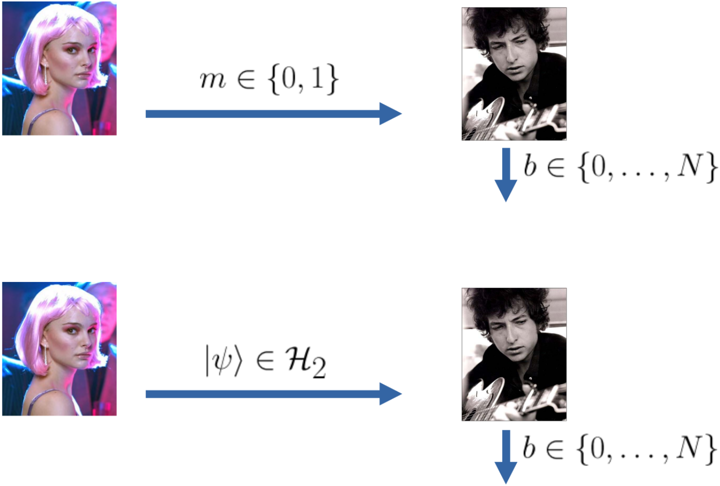
The prepare and measure scenario



$m \in \{0, 1\}$

$b \in \{0, \dots, N\}$

The prepare and measure scenario



The prepare and measure scenario

Qubits are better!

$$0 \mapsto |0\rangle, \quad 1 \mapsto |1\rangle$$

The prepare and measure scenario

Qubits are strictly better!!

Random Access Coding

$x_0, x_1 \in \{0,1\}$

$m \in \{0,1\}$

$y \in \{0,1\}$

$b \in \{0,1\}$

win if $b = x_y$

Random Access Coding

$x_0, x_1 \in \{0,1\}$

$y \in \{0,1\}$

$m \in \{0,1\}$

$b \in \{0,1\}$

$$\text{win if } b = x_y$$

$$p_{\text{classic}} \leq \frac{3}{4}$$

Random Access Coding

$x_0, x_1 \in \{0,1\}$

$|\psi\rangle \in \mathcal{H}_2$

$y \in \{0,1\}$

$b \in \{0,1\}$

win if $b = x_y$

Random Access Coding

$x_0, x_1 \in \{0,1\}$

$|\psi\rangle \in \mathcal{H}_2$

$y \in \{0,1\}$

$b \in \{0,1\}$

$$\text{win if } b = x_y$$

$$p_{\text{quantum}} \leq \frac{2+\sqrt{2}}{4} \approx 85\%$$

Random Access Coding

What if Alice sends 2 bits?

Prepare-and-Measure

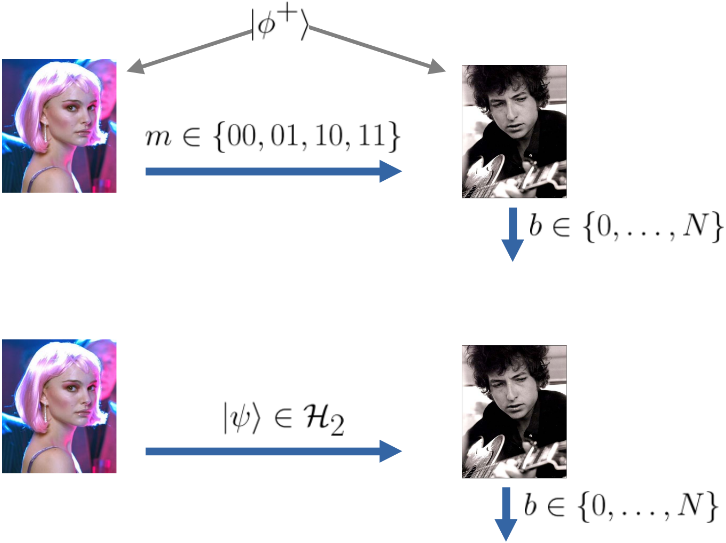
$m \in \{00, 01, 10, 11\}$

$b \in \{0, \ldots, N\}$

$|\psi\rangle \in \mathcal{H}_2$

$b \in \{0, \ldots, N\}$

Quantum teleportation



No extra resource?

$m \in \{00, 01, 10, 11\}$

$b \in \{0, \ldots, N\}$

$|\psi\rangle \in \mathcal{H}_2$

$b \in \{0, \ldots, N\}$

Qubit simulation requires unlimited shared randomness

$m \in \{0, \ldots, d_C - 1\}$

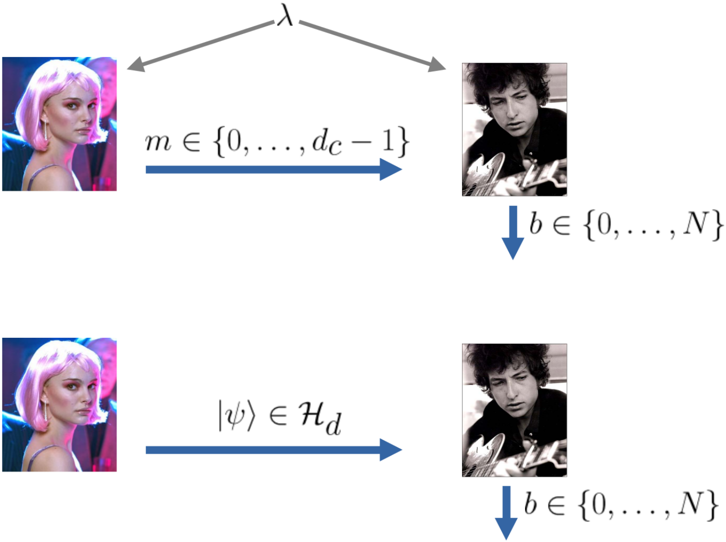
$b \in \{0, \ldots, N\}$

$|\psi\rangle \in \mathcal{H}_2$

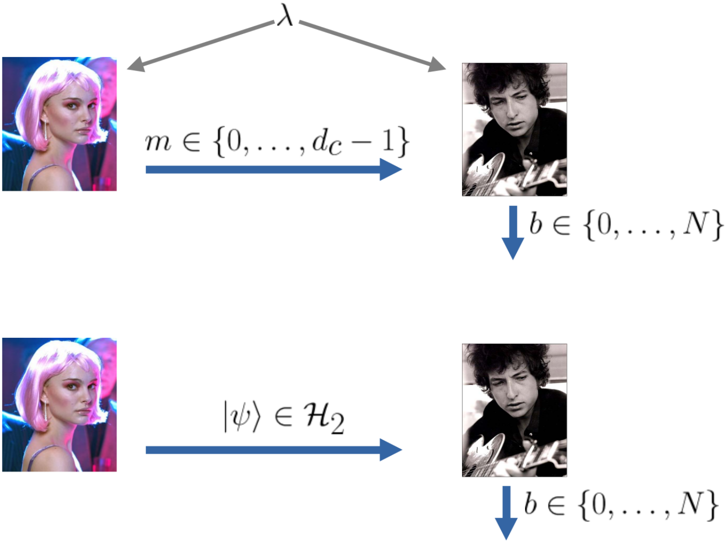
$b \in \{0, \ldots, N\}$

Massar, Bacon, Cerf, and Cleve, PRA (2001)

Prepare and Measure with Shared Randomness



The problem:


λ
$m \in \{0, \ldots, d_C - 1\}$
$b \in \{0, \ldots, N\}$
$|\psi\rangle \in \mathcal{H}_2$
$b \in \{0, \ldots, N\}$

Previous status:

- Toner and Bacon (PRL, 2003): If Bob performs projective measurements, 2bits+SR can simulate a qubit!

Previous status:

- Toner and Bacon (PRL, 2003): If Bob performs projective measurements, 2bits+SR can simulate a qubit!
- POVMs?

Previous status:

- Toner and Bacon (PRL, 2003): If Bob performs projective measurements, 2bits+SR can simulate a qubit!
- POVMs?
- POVMs allow us to go beyond dichotomic scenarios

Previous status:

- Toner and Bacon (PRL, 2003): If Bob performs projective measurements, 2bits+SR can simulate a qubit!
- POVMs?
- POVMs allow us to go beyond dichotomic scenarios
- POVMs do exist!

Previous status:

- Toner and Bacon (PRL, 2003): If Bob performs projective measurements, 2bits+SR can simulate a qubit!
- POVMs?
- POVMs allow us to go beyond dichotomic scenarios
- POVMs do exist!
- POVMs are known to outperform projective measurements in several similar cases...

Previous status:

- Toner and Bacon (PRL, 2003): If Bob performs projective measurements, 2bits+SR can simulate a qubit!
- POVMs?
- POVMs allow us to go beyond dichotomic scenarios
- POVMs do exist!
- POVMs are known to outperform projective measurements in several similar cases. . .
- Unambiguous state discrimination
  Unbounded randomness certification, PRA 95, 020102(R) (2017)
  Several PM tasks, PRA 92, 042117 (2015)
  etc. . .

Previous status:

- Toner and Bacon (PRL, 2003): If Bob performs projective measurements, 2bits+SR can simulate a qubit!
- POVMs?
- POVMs allow us to go beyond dichotomic scenarios
- POVMs do exist!
- POVMs are known to outperform projective measurements in several similar cases. . .
- Unambiguous state discrimination
  Unbounded randomness certification, PRA 95, 020102(R) (2017)
  Several PM tasks, PRA 92, 042117 (2015)
  etc. . .
- How about 1trit+SR?

Previous status:

- Toner and Bacon (PRL, 2003): If Bob performs projective measurements, 2bits+SR can simulate a qubit!
- POVMs?
- POVMs allow us to go beyond dichotomic scenarios
- POVMs do exist!
- POVMs are known to outperform projective measurements in several similar cases. . .
- Unambiguous state discrimination
  Unbounded randomness certification, PRA 95, 020102(R) (2017)
  Several PM tasks, PRA 92, 042117 (2015)
  etc. . .
- How about 1trit+SR?
- Buhrman, Cleve, Massar, de Wolf, Rev. Mod. Phys. (2010). Non-locality and communication complexity
  Many results, but not much about minimal worst case scenarios. . .

Our goal:

- 1: Analyse the trit vs Qubit case in detail

Our goal:

- 1: Analyse the trit vs Qubit case in detail
- 2: Understand the power and limitations of POVMs

1: Trits *vs* Qubits

1: Trits *vs* Qubits

For some tasks, a trit is better than qubit

1: Trits *vs* Qubits

$x_0, x_1 \in \{0, 1\}$

$m \in \{0, 1, 2\}$

$y \in \{0, 1\}$

$b \in \{0, 1\}$

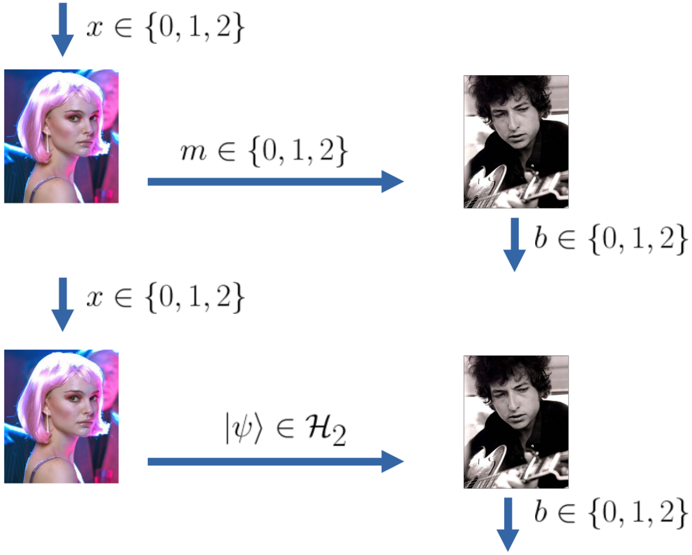
$$\text{win if } b = x_y$$

$$p_{\text{trit}} \leq \frac{7}{8}$$

1: Trits *vs* Qubits

For some tasks, a trit is better than qubit
(Holevo bound!)

1: Trits *vs* Qubits



1: Question?

Are trits strictly better than qubits?

RESULT 1

Are trits strictly better than qubits?

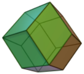
No!

RESULT 1

For some tasks, a qubit is better than trit

RESULT 1

$|\psi_m\rangle \in$ 

$M_y \in$ 

$|\psi\rangle \in \mathcal{H}_2$

$b \in \{0,1\}$

$$\text{prob}\left(b \,\middle|\, |\psi_m\rangle, M_y\right) = \text{Tr}\left(|\psi_m\rangle\langle\psi_m|\, M_{b|y}\right)$$

RESULT 1

$x \in \{0, \ldots, 5\}$

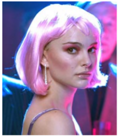
$m \in \{0, 1, 2\}$

$y \in \{0, \ldots, 23\}$

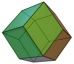
Cannot simulate the qubit stragy

$b \in \{0, 1\}$

$|\psi_m\rangle \in$ 



$|\psi\rangle \in \mathcal{H}_2$

$M_y \in$ 

$b \in \{0, 1\}$

RESULT 1 methods

- Recognise that the problem is a linear program (even with robustness considered)

RESULT 1 methods

- Recognise that the problem is a linear program (even with robustness considered)
- Optimisation trick to reduce complexity

RESULT 1 methods

- Recognise that the problem is a linear program (even with robustness considered)
- Optimisation trick to reduce complexity
- Find a PM task via the dual problem

RESULT 1 methods

- Recognise that the problem is a linear program
  (even with robustness considered)
- Optimisation trick to reduce complexity
- Find a PM task via the dual problem
- Extract a computer-assisted proof
  (as in Bavaresco, Murao, Quintino, PRL 127, 200504 (2021))

RESULT 1 methods

- Recognise that the problem is a linear program (even with robustness considered)
- Optimisation trick to reduce complexity
- Find a PM task via the dual problem
- Extract a computer-assisted proof (as in Bavaresco, Murao, Quintino, PRL 127, 200504 (2021))
- Various examples, minimal: 6 preparations, 11 measurements

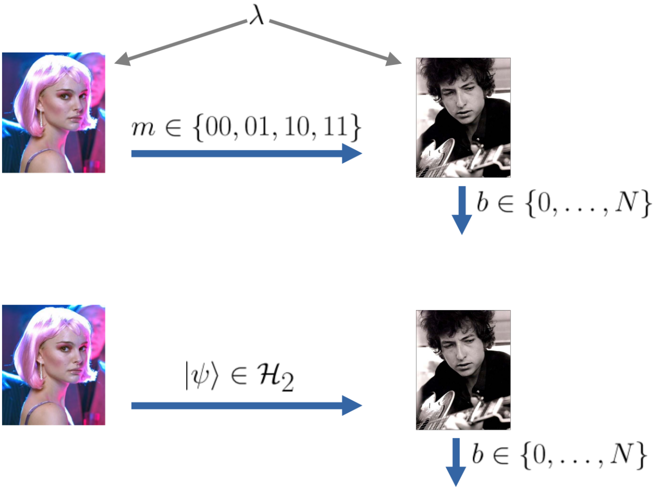
2: POVMs?

2: POVMs?

Are 2bits strictly better than 1qubit?

2: POVMs?

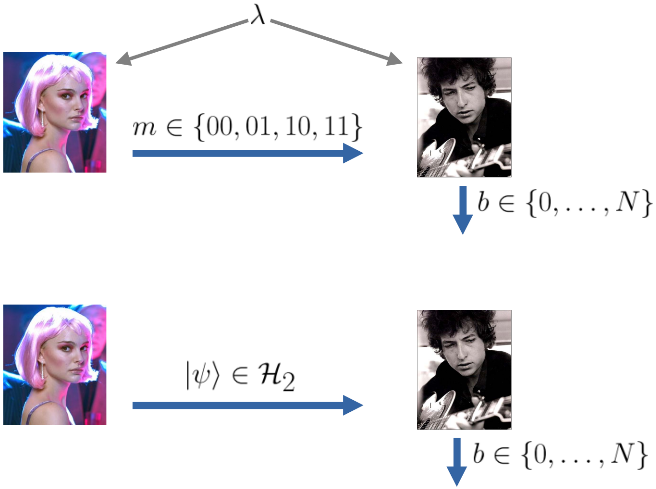
Are 2bits strictly better than 1qubit?



2: POVMs?

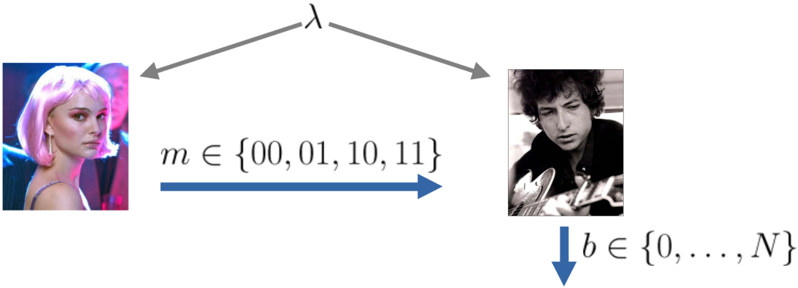
Are 2bits strictly better than 1qubit?

YES!



RESULT 2

2bits+SR is strictly better than qubits!

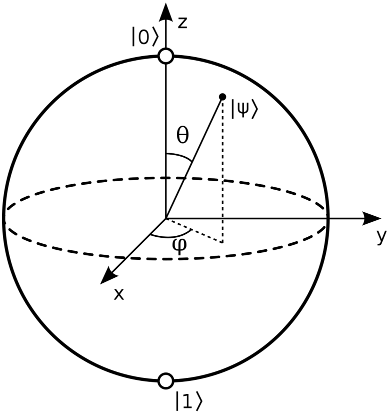


RESULT 2

Proof: Explicit recipe for classical simulation

RESULT 2 methods

use the Bloch sphere to states and POVM elements



RESULT 2 methods

- $\lambda := (\vec{\lambda}_1, \vec{\lambda}_2)$, random vectors on the sphere

RESULT 2 methods

- $\lambda := (\vec{\lambda}_1, \vec{\lambda}_2)$, random vectors on the sphere
- Instead of $\rho = \frac{1}{2}(I + \vec{x} \cdot \vec{\sigma})$, Alice sends $c_1 = H(\vec{x} \cdot \vec{\lambda}_1)$ and $c_2 = H(\vec{x} \cdot \vec{\lambda}_2)$
  Heaviside: $H(x) = 1$ if $x \geq 0$, $H(x) = 0$ if $x < 0$).

RESULT 2 methods

- $\lambda := (\vec{\lambda}_1, \vec{\lambda}_2)$, random vectors on the sphere
- Instead of $\rho = \frac{1}{2}(I + \vec{x} \cdot \vec{\sigma})$, Alice sends $c_1 = H(\vec{x} \cdot \vec{\lambda}_1)$ and $c_2 = H(\vec{x} \cdot \vec{\lambda}_2)$
  Heaviside: $H(x) = 1$ if $x \geq 0$, $H(x) = 0$ if $x < 0$).
- Bob finds the Bloch vectors for the POVM elements, $B_b = p_b(I + \vec{y}_b \cdot \vec{\sigma})$ then sets $\vec{\lambda} := (-1)^{1+c_1}\vec{\lambda}_1$ when $|\vec{\lambda}_1 \cdot \vec{y}_b| \geq |\vec{\lambda}_2 \cdot \vec{y}_b|$ and $\vec{\lambda} := (-1)^{1+c_2}\vec{\lambda}_2$ otherwise.

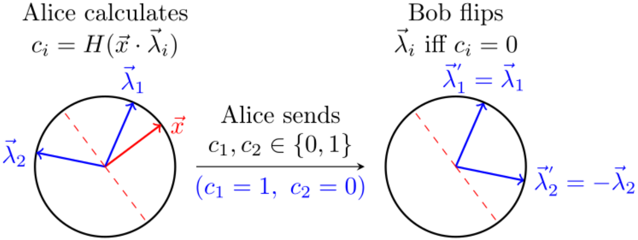
RESULT 2 methods

- $\lambda := (\vec{\lambda}_1, \vec{\lambda}_2)$, random vectors on the sphere
- Instead of $\rho = \frac{1}{2}(I + \vec{x} \cdot \vec{\sigma})$, Alice sends $c_1 = H(\vec{x} \cdot \vec{\lambda}_1)$ and $c_2 = H(\vec{x} \cdot \vec{\lambda}_2)$
  Heaviside: $H(x) = 1$ if $x \geq 0$, $H(x) = 0$ if $x < 0$).
- Bob finds the Bloch vectors for the POVM elements, $B_b = p_b(I + \vec{y}_b \cdot \vec{\sigma})$ then sets $\vec{\lambda} := (-1)^{1+c_1}\vec{\lambda}_1$ when $|\vec{\lambda}_1 \cdot \vec{y}_b| \geq |\vec{\lambda}_2 \cdot \vec{y}_b|$ and $\vec{\lambda} := (-1)^{1+c_2}\vec{\lambda}_2$ otherwise.
- Finally, Bob outputs b with probability:

$$p(b|\{\vec{y}_b\}_b, \lambda) = \frac{p_b\ \Theta(\vec{y}_b \cdot \vec{\lambda})}{\sum_{j=1}^{n}\ p_j\ \Theta(\vec{y}_j \cdot \vec{\lambda})}$$

$$\Theta(x) := \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

RESULT 2 methods



RESULT 2 methods

Why it works?

RESULT 2 methods

Why it works?

Well...

$$\forall \text{ qubit } \rho, \forall \text{ POVM } \{M_b\}$$

$$\int_\lambda \mathrm{d}\lambda \pi(\lambda) \sum_{c=1}^{4} p_A(c|\rho,\lambda) p_B(b|\{M_b\},c,\lambda) = \mathrm{tr}(\rho\, M_b)$$

RESULT 2 methods

Why it works?

Lemma 1. *Given two normalized vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$ on the unit sphere S_2, it holds that:*

$$\frac{1}{\pi} \int_{S_2} H(\vec{x} \cdot \vec{\lambda}) \cdot \ \Theta(\vec{y} \cdot \vec{\lambda}) \ \mathrm{d}\vec{\lambda} = \frac{1}{2}(1 + \vec{x} \cdot \vec{y}),$$

where $H(z)$ is the Heaviside function ($H(z) = 1$ if $z \geq 0$ and $H(z) = 0$ if $z < 0$) and $\Theta(z) := H(z) \cdot z$.

RESULT 2 extra

- The fraction of rounds in which Alice is communicating only a single bit to Bob has measure zero.

RESULT 2 extra

- The fraction of rounds in which Alice is communicating only a single bit to Bob has measure zero.
- This holds for any protocol that exactly simulates any qubit strategy in a prepare-and-measure scenario.

Implications to Bell Nonlocality

Implications to Bell Nonlocality

$x \in \{0, \ldots, N\}$ $\qquad$ $y \in \{0, \ldots, N\}$

$|\phi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2$

$a \in \{0, \ldots, N\}$ $\qquad$ $b \in \{0, \ldots, N\}$

Implications to Bell Nonlocality

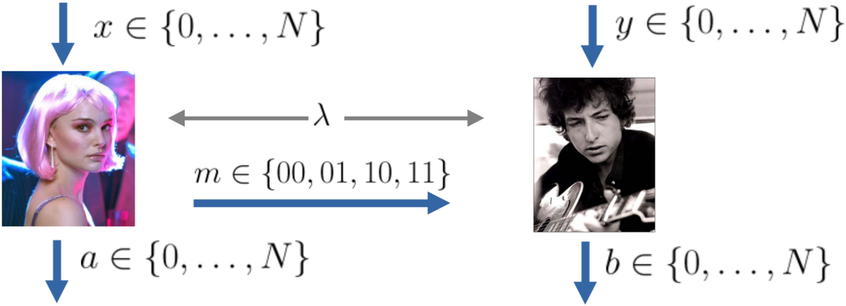
$x \in \{0, \dots, N\}$ $\qquad$ $y \in \{0, \dots, N\}$

$|\phi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2$

$a \in \{0, \dots, N\}$ $\qquad$ $b \in \{0, \dots, N\}$

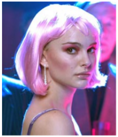
$$p(ab|xy) \neq \sum_\lambda \pi(\lambda) p(a|x,\lambda) p(b|y,\lambda)$$

Implications to Bell Nonlocality



Implications to Bell Nonlocality

$x \in \{0, \ldots, N\}$ $y \in \{0, \ldots, N\}$

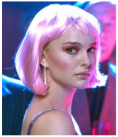


λ

$m \in \{00, 01, 10, 11\}$

$a \in \{0, \ldots, N\}$ $b \in \{0, \ldots, N\}$

$x \in \{0, \ldots, N\}$ $y \in \{0, \ldots, N\}$

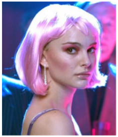


$|\phi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2$

$a \in \{0, \ldots, N\}$ $b \in \{0, \ldots, N\}$

Implications to Bell Nonlocality

$x \in \{0, \ldots, N\}$ $y \in \{0, \ldots, N\}$



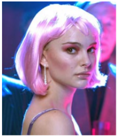
λ

$m \in \{00, 01, 10, 11\}$

BETTER

$a \in \{0, \ldots, N\}$ $b \in \{0, \ldots, N\}$

$x \in \{0, \ldots, N\}$ $y \in \{0, \ldots, N\}$



$|\phi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2$

$a \in \{0, \ldots, N\}$ $b \in \{0, \ldots, N\}$

Outlook

- 2bits + SR $>$ 1qbit (PM scenario)

Outlook

- 2bits + SR > 1qbit (PM scenario)
- even with general POVM measurements!

Outlook

- 2bits + SR > 1qbit (PM scenario)
- even with general POVM measurements!
- No hierarchy between 1trit and 1qubit (PM scenario)

Outlook

- 2bits + SR > 1qbit (PM scenario)
- even with general POVM measurements!
- No hierarchy between 1trit and 1qubit (PM scenario)
- 2 bits of communication > two-qubit states (Bell scenario)

Outlook

- 2bits + SR > 1qbit (PM scenario)
- even with general POVM measurements!
- No hierarchy between 1trit and 1qubit (PM scenario)
- 2 bits of communication > two-qubit states (Bell scenario)
- even with general POVM measurements!

Open problems

- How about qutrits??

Open problems

- How about qutrits??
- Not even clear if it can be done with finite classical communication. . . (even in the projective case)

Open problems

- How about qutrits??
- Not even clear if it can be done with finite classical communication. . . (even in the projective case)
- Prepare-and-measure models and Bell with communication models?

Open problems

- How about qutrits??
- Not even clear if it can be done with finite classical communication. . . (even in the projective case)
- Prepare-and-measure models and Bell with communication models?
- Minimal models for Bell with communication are different!

Open problems

- How about qutrits??
- Not even clear if it can be done with finite classical communication. . . (even in the projective case)
- Prepare-and-measure models and Bell with communication models?
- Minimal models for Bell with communication are different!
- *e.g.*, One trit is enough to simulate two-qubit Bell correlations

  The minimal communication cost for simulating entangled qubits, arXiv:2207.12457
  M. Renner, M.T. Quintino

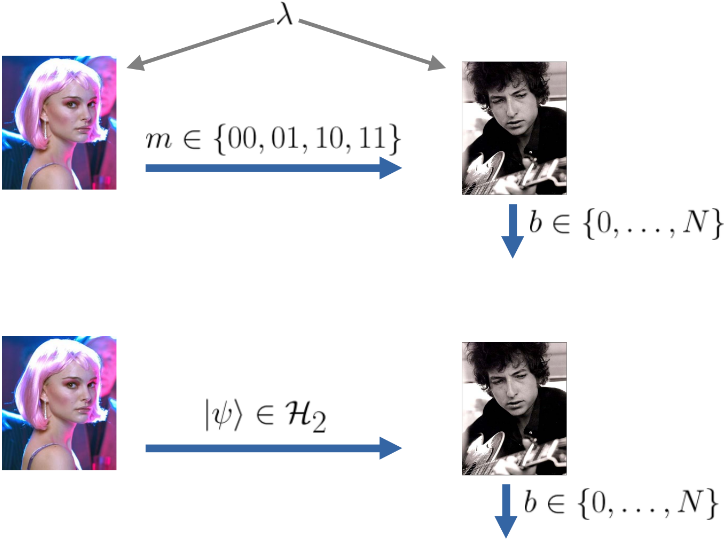
Open problems

- How about qutrits??
- Not even clear if it can be done with finite classical communication. . . (even in the projective case)
- Prepare-and-measure models and Bell with communication models?
- Minimal models for Bell with communication are different!
- *e.g.*, One trit is enough to simulate two-qubit Bell correlations

  The minimal communication cost for simulating entangled qubits, arXiv:2207.12457
  M. Renner, M.T. Quintino
- *e.g.*, One bit might be enough to simulate two-qubit Bell correlations

  Classical Simulation of Two-Qubit Entangled States with One Bit of Communication, arXiv:2305.19935
  P. Sidajaya, A. D. Lim, B. Yu, V. Scarani

Thank you!


λ
$m \in \{00, 01, 10, 11\}$
$b \in \{0, \ldots, N\}$
$|\psi\rangle \in \mathcal{H}_2$
$b \in \{0, \ldots, N\}$