Transforming and discriminating quantum operations using higher-order methods

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Quantum state transformations

$$|\psi_{\text{in}}\rangle \mapsto |\psi_{\text{out}}\rangle$$

Quantum state transformations

$$|\psi_{\text{in}}\rangle \mapsto |\psi_{\text{out}}\rangle = U|\psi_{\text{in}}\rangle$$

$$U^\dagger U = I$$

Quantum state transformations

$$\rho_{\mathrm{in}} \mapsto \rho_{\mathrm{out}}$$

Quantum state transformations

$$\rho_{\text{in}} \mapsto \rho_{\text{out}} = \widetilde{\Lambda}(\rho_{\text{in}})$$

$\widetilde{\Lambda}$ is CPTP

Quantum operation transformations

Can we transform quantum operations??

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$$U_{\text{in}} \mapsto U_{\text{out}}$$

$$\widetilde{\Lambda\text{in}} \mapsto \widetilde{\Lambda_{\text{out}}}$$

"Quantum" unitary inversion

$$U_d \mapsto U_d^{-1}$$

The universal/unknown paradigm

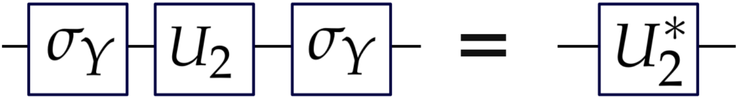
$$\sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \sigma_Z^{-1}$$

What do we want?

What do we want?

Ideally. . .

Something like this:



Phys. Rev. Research (2019)
J. Miyazaki, A. Soeda, and M. Murao

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  G. Chiribella and D. Ebler, New Journal of Physics (2016)

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- Exact
- Possible?
- Optimal average fidelity: $F_{max} = \frac{2}{d^2}$
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- $F_{max} < 1 \implies$ Impossible...

What do we want?

Probabilistic heralded?

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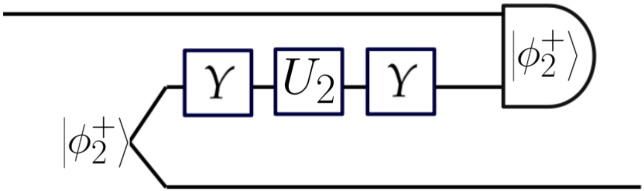
Probabilistic heralded?

For qubits, Possible!

Explicit construction



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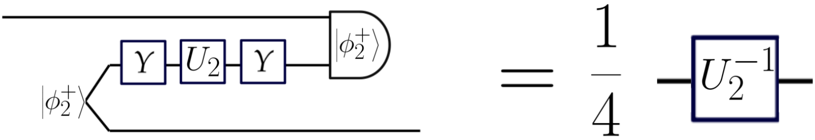
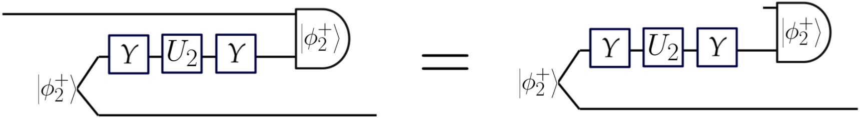


Delayed input state protocols



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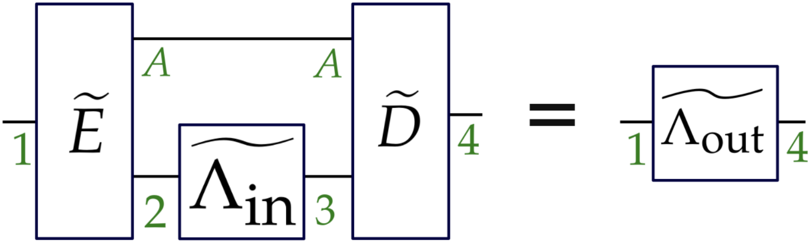
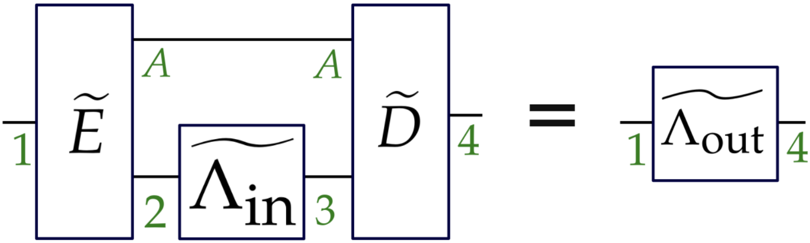
- Higher-order operations and supermaps!

Superchannels



The most general quantum superchannel?

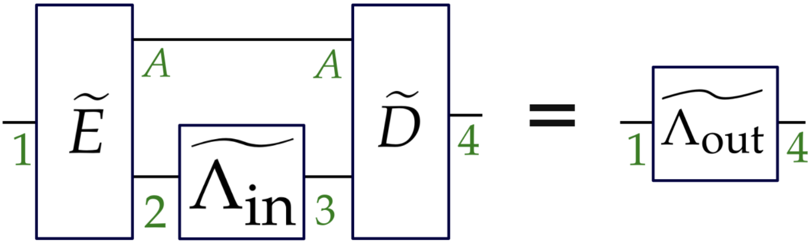
Superchannels



$$\tilde{\tilde{S}} : [\mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3)] \to [\mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_4)]$$

- $\tilde{\tilde{S}}$ is a *linear supermap*

Superchannels



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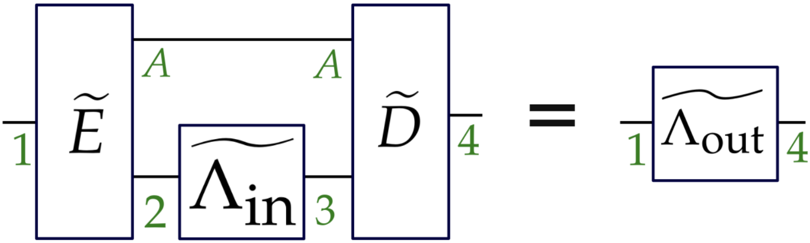

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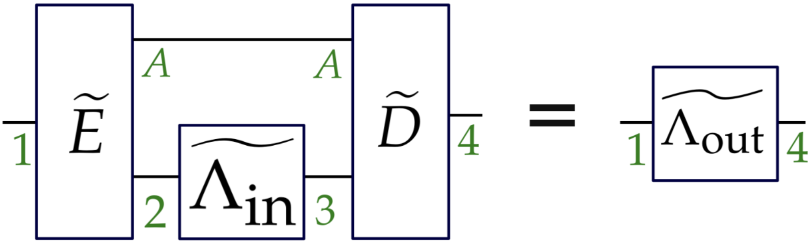
- $\widetilde{\widetilde{S}}$ may be applied into part of channel (completely CP preserving)

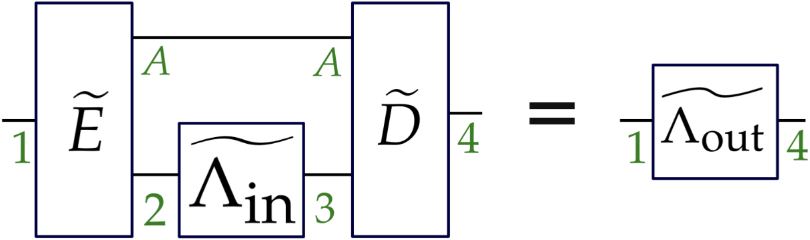
Superchannels



$$\widetilde{\widetilde{S}}(\widetilde{\Lambda_{\rm in}}) = \mathrm{tr}_A \left(\widetilde{D} \circ \left(\widetilde{\Lambda_{\rm in}} \otimes \widetilde{I_A} \right) \circ \widetilde{E} \right)$$

G. Chiribella, G. M. D'Ariano, and P. Perinotti EPL (2008)
K. Życzkowski J. Phys. A 41, 355302-23 (2008)
G. Gutoski and J. Watrous Proceedings of STOC (2007)

Superchannels



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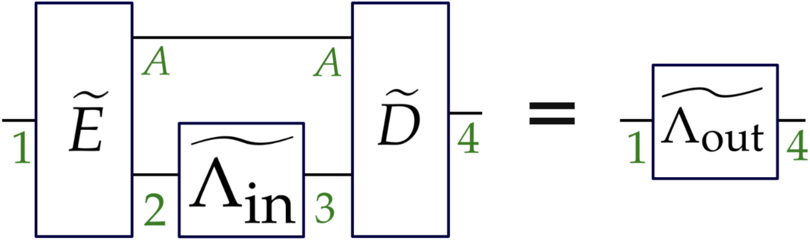
Superchannels



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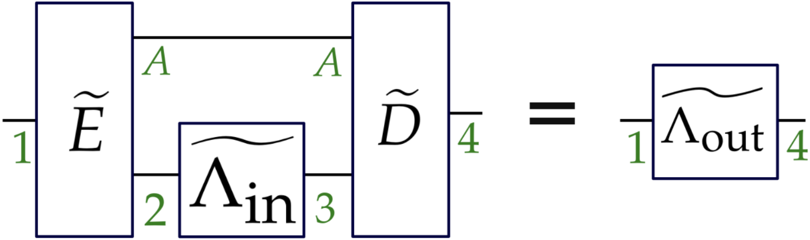

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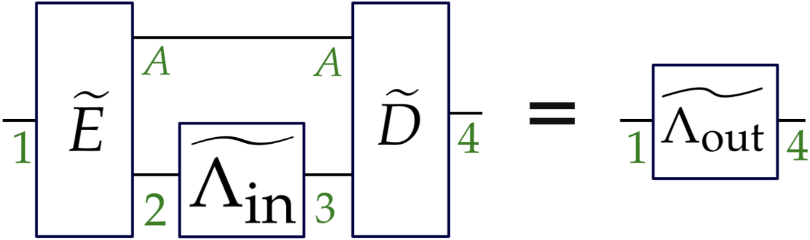

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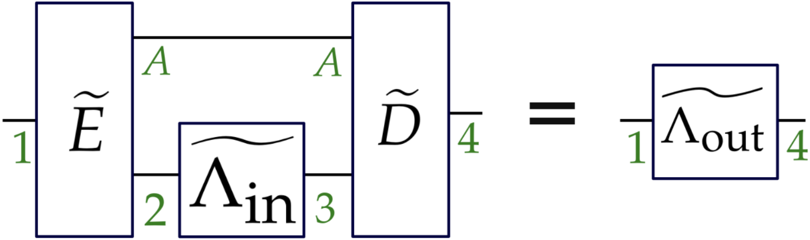

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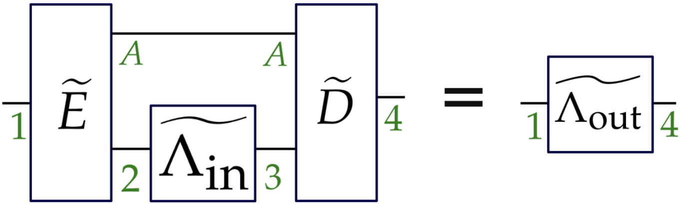
Superchannels



$S \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes H_4)$ is a superchannel iff

$$S \geq 0$$

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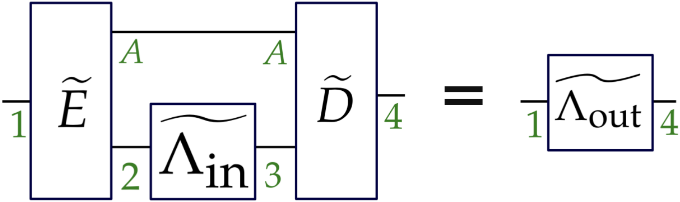

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Affine and positive semidefinite constraints $\implies$ SDP!!

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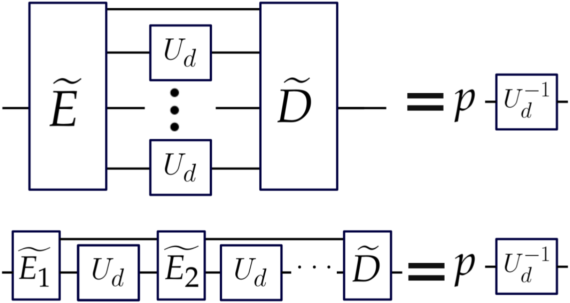

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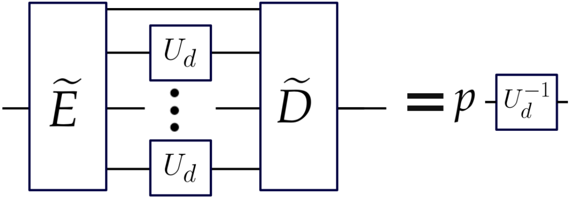
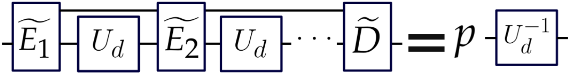
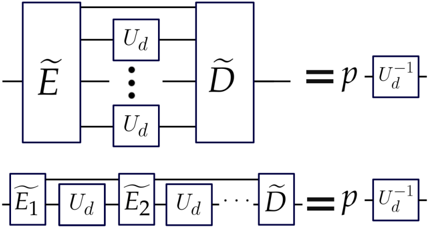




(Quantum combs, channel with memory, quantum strategy, quantum superchannels with multiple inputs)

Results



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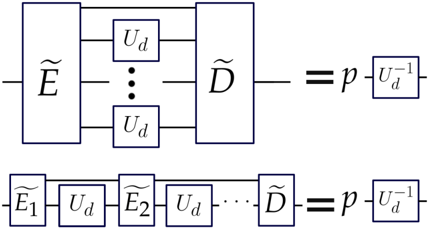

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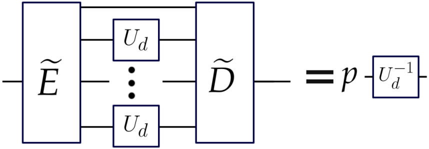

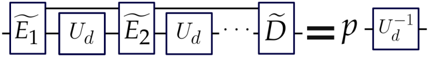

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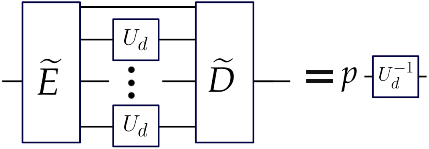

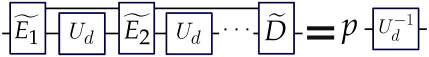

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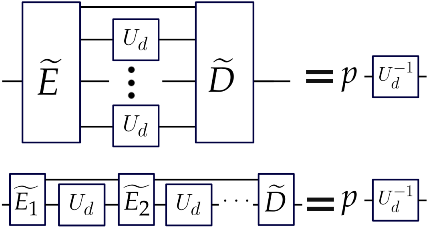

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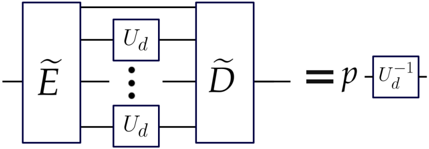

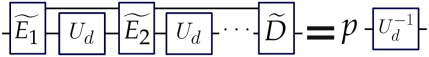

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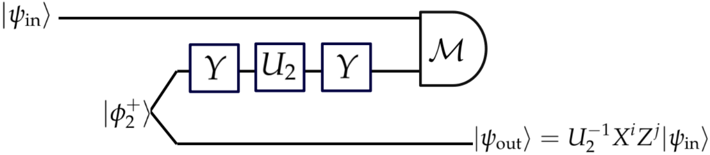
Q. Dong, A. Shimbo, A. Soeda, M. Murao
PRA (2019), PRL (2019)

Qubit adaptive circuit



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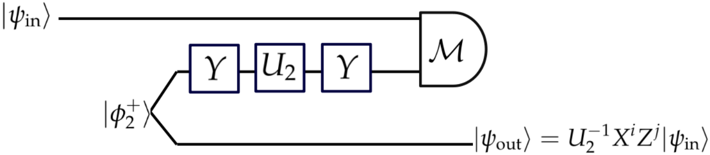
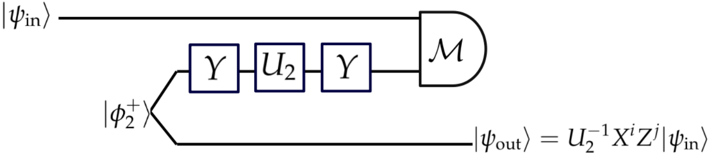


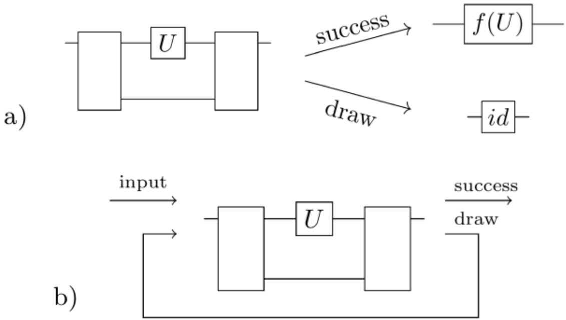
If we fail, we "destroy" the unknown input state...
then we cannot re-iterate this protocol...

Qubit adaptive circuit



But well...
if we can use U_2 once more:
apply $X^{-i}Z^{-j}U_2$ to recover the input state!

Success or draw



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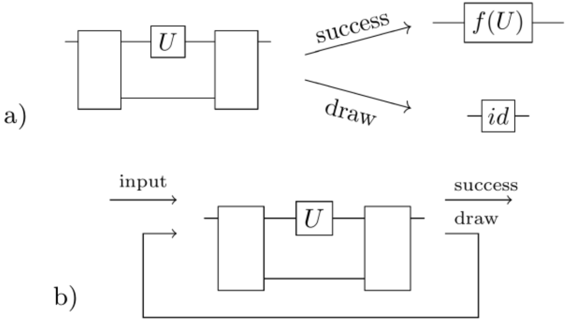


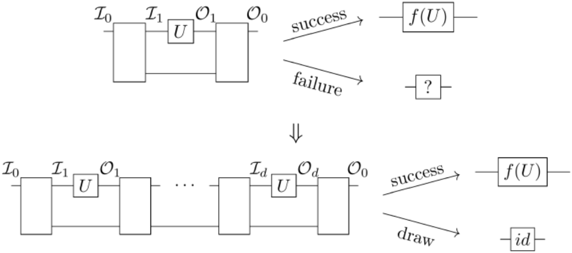
With k uses, this approach leads to a success probability of

$$p_s = 1 - (1 - p_{\text{draw}})^k$$

Success or draw

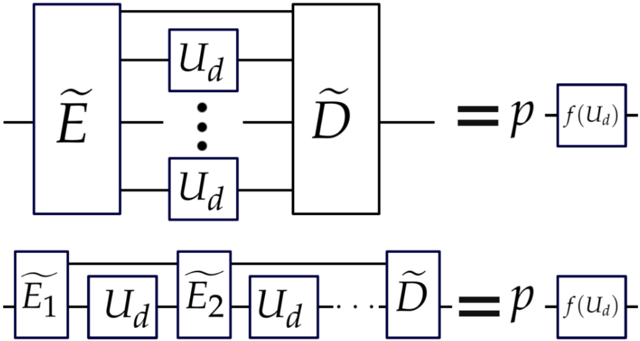
Theorem

Success or draw is always possible!



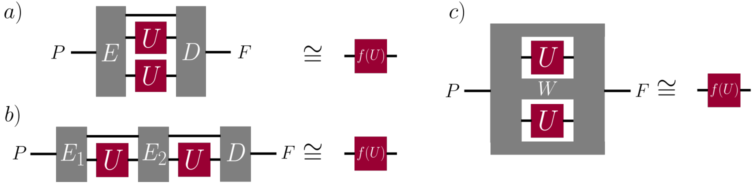
Q. Dong, **MTQ**, A. Soeda, M. Murao
PRL (2021)

Arbitrary functions $f(U_d)$



Deterministic protocols?

- Deterministic non-exact is also interesting!



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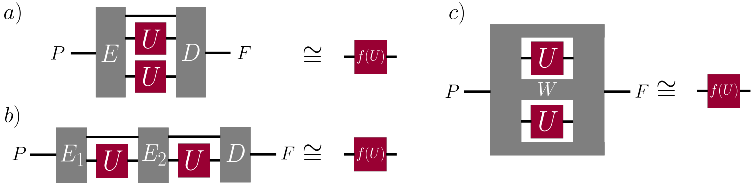

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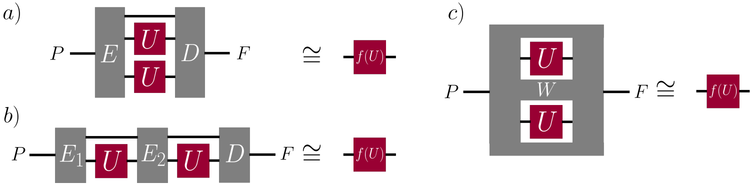

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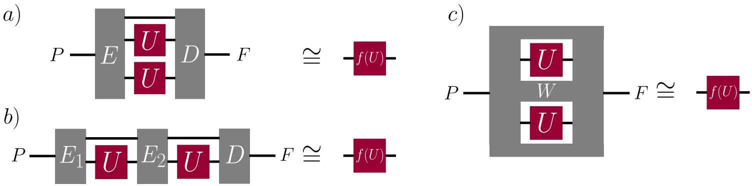
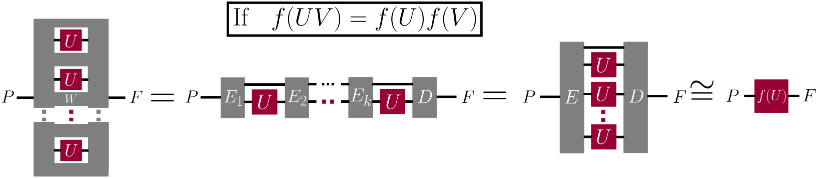


- Average channel fidelity (over the Haar measure)
- Maximal white noise visibility
- Parallel inversion = Parallel transposition = Unitary estimation

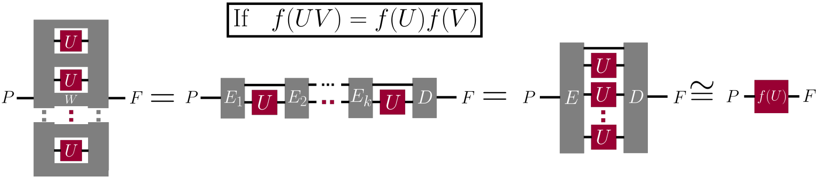
MTQ, D. Ebler, Quantum 2022

Solving the homomorphic case

- 

A. Bisio, G.M. D'Ariano, P. Perinotti, M. Sedlak
Physics Letters A (2014)

Solving the homomorphic case

- 

A. Bisio, G.M. D'Ariano, P. Perinotti, M. Sedlak
Physics Letters A (2014)

- *e.g.*, $(UV)^* = U^*V^*$, and cloning

Solving the homomorphic case

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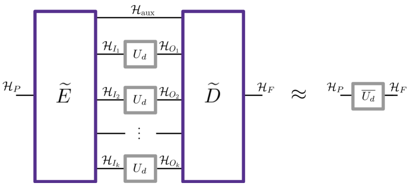

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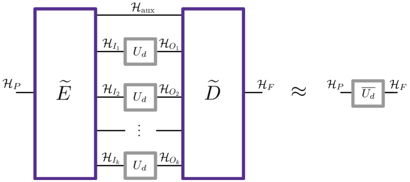

- Step 3: use SDP duality and group theoretic results on Young-Tableau to prove $F \leq \frac{k+1}{d(d-1)}$

IEEE Transactions on Information Theory
D. Ebler, M. Horodecki, M. Marciniak, T. Młynik, **MTQ**, M. Studziński

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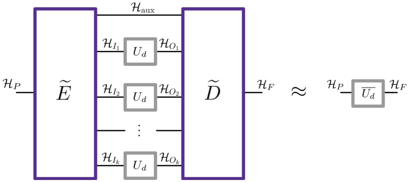

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- Step 4: prove that if $k < d - 1$, $p = 0$
  PRA
  MTQ, Q. Dong, A. Shimbo, A. Soeda, M. Murao

Progress on the anti-homomorphic case

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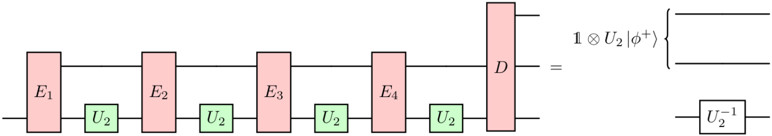
- But, there are still various open questions...
- For instance: when $d = 2$ we have

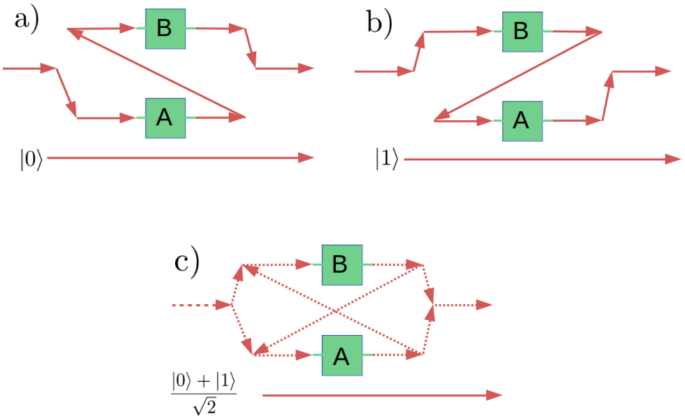


PRL, S. Yoshida, A. Soeda, M. Murao

More general superchannels?

Can we go beyond sequential quantum circuits?

Quantum Switch



Quantum computations without definite causal structure
G. Chiribella, G. M. D'Ariano, P. Perinotti, B. Valiron
PRA 2013

More general superchannels?

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Process Matrices! (May have an indefinite causal order)
G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, PRA (2013)
O. Oreshkov, F. Costa, and Č. Brukner, Nature Communications (2012)

Process matrices

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Summary of results

Deterministic unitary inversion: $U^{\otimes k} \overset{\approx}{\longmapsto} U^{-1}$

	Parallel	Sequential	General
$k=1$	$F=\frac{2}{d^2}$	$F=\frac{2}{d^2}$	$F=\frac{2}{d^2}$
$d=2$	$F=1-\sin^2\left(\frac{\pi}{k+3}\right)$	?	?
(d,k)	$\lVert M_{\text{est}} \rVert$	?	?
$k\to\infty$	$F\approx 1-\frac{1}{k^2}$	$1-\frac{1}{e^k}\leq F\leq ?$	?
Estimation	$F_{\text{par}}=F_{\text{est}}$	N/A	N/A
PBT	?	N/A	N/A
$k\leq d-1$	$F=\frac{k+1}{d^2}$	$F=\frac{k+1}{d^2}$	$F=\frac{k+1}{d^2}$
$d=2,k=4$	$F=1-\sin^2\left(\frac{\pi}{7}\right)$	$F=1$	$F=1$
$d=2,k=2$	$F=1-\sin^2\left(\frac{\pi}{5}\right)$	$F=\frac{3}{4}$	$F\approx 0.8249$
$d=3,k=2$	$F=\frac{1}{3}$	$F=\frac{1}{3}$	$F=\frac{1}{3}$

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Estimation	$F_{\text{par}}=F_{\text{est}}$	N/A	N/A
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Probabilistic unitary inversion: $U^{\otimes k} \mapsto p\,U^{-1}$

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$k=1$	$p=\frac{1}{d^2}$	$p=\frac{1}{d^2}$	$p=\frac{1}{d^2}$
$d=2$	$p=1-\frac{3}{k+3}$	?	?
(d,k)	?	?	?
$k\to\infty$	$p\approx 1-\frac{1}{k}$	$1-\frac{1}{e^k}\le p\le ?$	?
Store-retrieve	?	N/A	N/A
PBT	?	N/A	N/A
$k\le d-1$	?	?	?
$d=2,k=4$	$p=\frac{4}{7}$	$p=1$	$p=1$
$d=2,k=2$	$p=\frac{2}{5}$	$p=\frac{3}{7}$	$p\approx 4/9$
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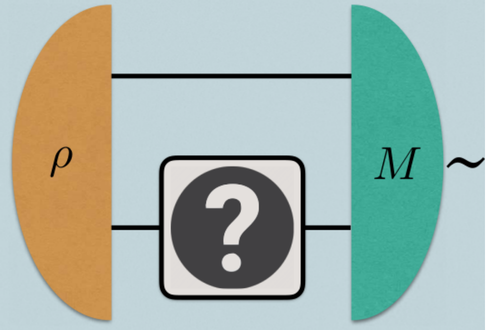
(d,k)	$p=1-\frac{d^2-1}{k+d^2-1}$	?	?
$k\to\infty$	$p\approx 1-\frac{1}{k}$	$1-\frac{1}{e^k}\le p\le ?$	?
Store-retrieve	$p_{\text{par}}=p_{\text{s-r}}$	N/A	N/A
PBT	$p_{\text{par}}=p_{\text{pbt}}$	N/A	N/A
$k\le d-1$	$p=1-\frac{d^2-1}{k+d^2-1}$	?	?
$d=2,k=4$	$p=\frac{4}{7}$	$p=1$	$p=1$
$d=2,k=2$	$p=\frac{2}{5}$	$p=\frac{3}{7}$	$p\approx 4/9$
$d=3,k=2$	$p=\frac{2}{10}$	$p\approx\frac{2}{9}$	$p\approx\frac{2}{8}$

Measuring quantum operations

How can one perform a measurement on a quantum operation?

Measuring quantum operations



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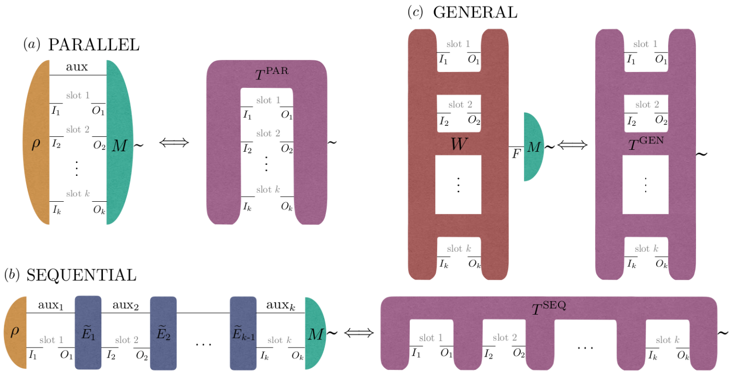

J. Bavaresco, M. Murao, MTQ PRL (2021)
J. Bavaresco, M. Murao, MTQ J. Math. Phys. (2022)

Quantum channel discrimination

- Two unitaries $\implies$ parallel is optimal

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- Almost always, we have a strict hierarchy!
  J. Bavaresco, M. Murao, **MTQ** PRL (2021)
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  J. Bavaresco, M. Murao, **MTQ** PRL (2021)
  J. Bavaresco, M. Murao, **MTQ** J. Math. Phys. (2022)

- SDP duality theory + indefinite causal order methods $\implies$ "universal upper bound" on unitary channel discrimination

Unitary channel discrimination

- For any ensemble,

$$\{p_i, U_i^{\otimes k}\}_{i=1}^{N}, \qquad U_i \in \mathrm{SU}(d)$$

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$$p_{succ} \leq \frac{1}{N} \frac{(k+d^2-1)!}{k!(d^2-1)!}$$

- When $\{U_i\}_{i=1}^N$ is a group k-design and $p_i = \frac{1}{N}$, this bound is saturated (by parallel strategies).

Final remarks

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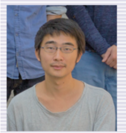
- Measuring between quantum operations
- New methods and new concepts
- Power and limitation of indefinite causality
- Can we have deterministic exact unitary inversion with finite sequential circuits?
- Asymptotic and practical advantages of indefinite causality?

Thank you!

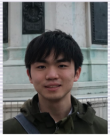
Mio Murao

Jisho Miyazaki

Atsushi Shimbo



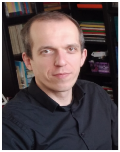
Qingxiuxiong Dong



Satoshi Yoshida

Akihito Soeda

Jessica Bavaresco



Michał Studziński

Michał Horodecki

Tomasz Młynik

Daniel Ebler

Thank you!