Measurement incompatibility and Bell nonlocality: from 1985 to 2022

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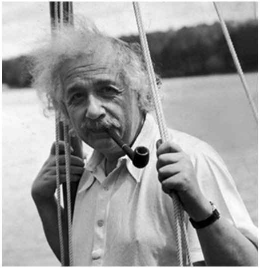
Federal Ministry
Education, Science
and Research

Measurement incompatibility

$$\Delta x \, \Delta p \geq \hbar/2$$

Quantum entanglement

$$\rho_{AB} \neq \int \pi(\lambda)\rho_A^\lambda \otimes \rho_B^\lambda \, \mathrm{d}\lambda$$



State+Measurement

$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i)$$

State+Measurement

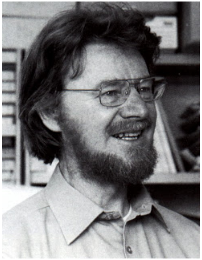
$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i), \quad |\langle i|\psi\rangle|^2$$

EPR steering

$$\sigma_{a|x} \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) \rho_\lambda$$

Bell nonlocality

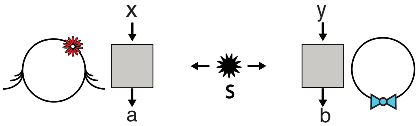
$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$



Relationship between all that

$$\|CHSH\| = \sqrt{4I + \|[A_1, A_2]\| \ \|[B_1, B_2]\|}$$

Bell Nonlocality



The Correlation/Anticorrelation Game

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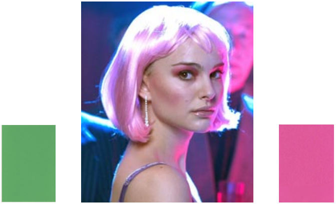
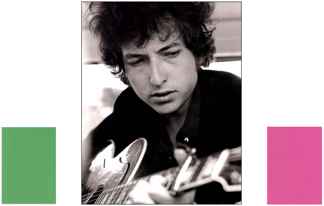
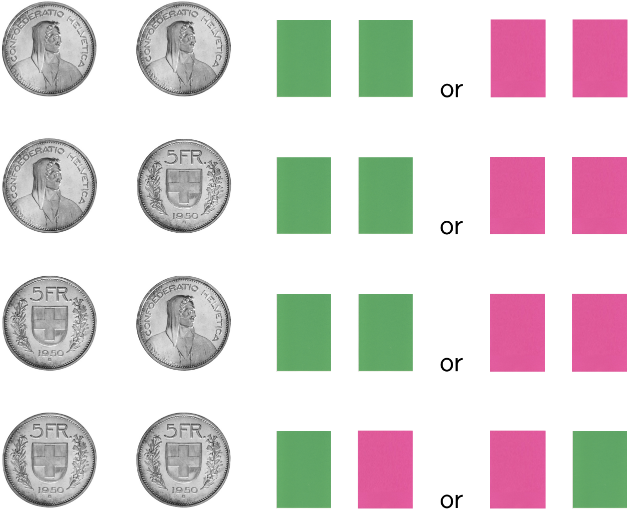




The Correlation/Anticorrelation Game

Winning Conditions



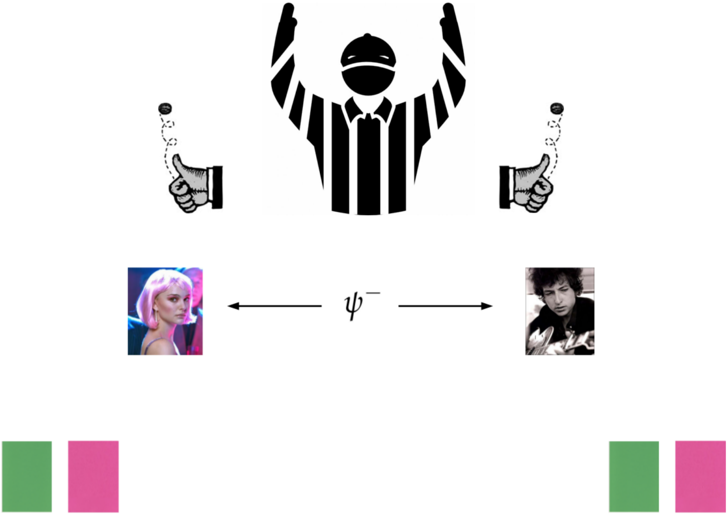
Best Strategy

Can Alice and Bob always win?

Best Strategy

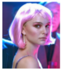
Best (classical) strategy wins with probability $\frac{3}{4}$

Quantum Strategy



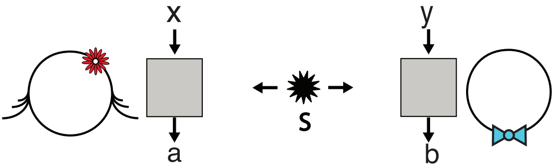
Quantum Strategy

$$p_q = \frac{2+\sqrt{2}}{4} \approx 0.8535$$

 $\longleftarrow \psi^- \longrightarrow$ 

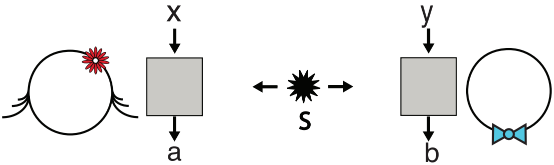
Bell nonlocality

$$p_{\mathrm{win}} = \sum_{abxy} \pi(x,y) V(ab|xy) p(ab|xy)$$



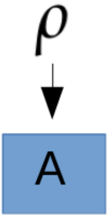
Bell nonlocality

$$p(ab|xy) \neq \sum_\lambda \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

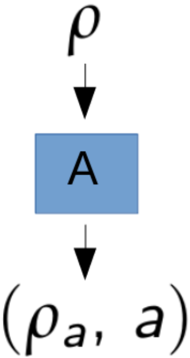


Quantum measurement

Quantum measurement



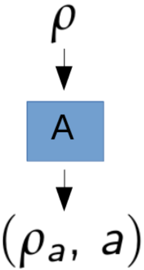
Quantum measurement



Quantum measurement

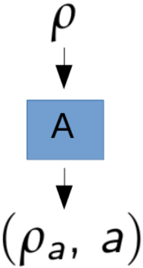
$$p(a|\rho, A) = \mathrm{tr}(\rho A_a)$$

$$A = \{A_a\}, \quad A_a \geq 0, \quad \sum_a A_a = I$$



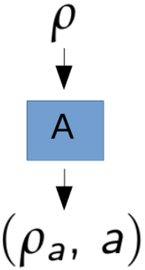
Quantum measurement

$$p(a|\rho, A) = \mathrm{tr}(\rho A_a), \quad \rho_a = A_a \text{ (if projective)}$$

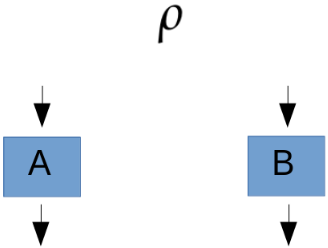


Quantum measurement

$$p(a|\rho, A) = \mathrm{tr}(\rho A_a), \quad \rho_a = \frac{\sqrt{A_a}\rho\sqrt{A_a}}{\rho A_a} \text{ (Lüders)}$$

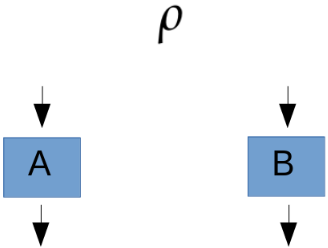


Measurement compatibility



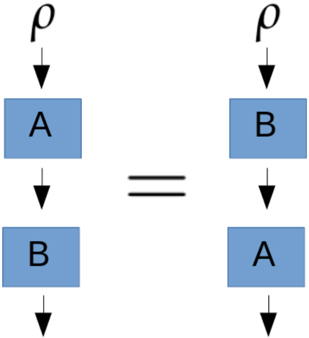
Measurement compatibility

Incompatible measurements are in the core of quantum theory



Measurement compatibility

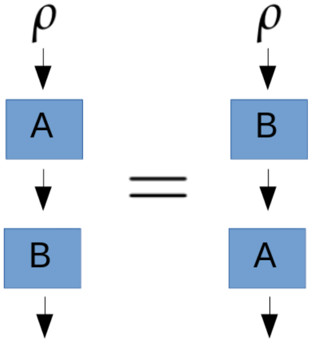
When the measurements commute:



Measurement compatibility

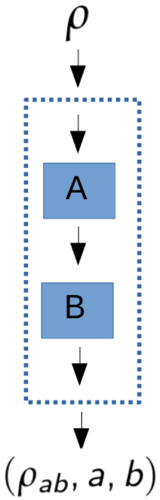
When the measurements commute:

$$[A_a, B_b] := A_a B_b - B_b A_a = 0$$



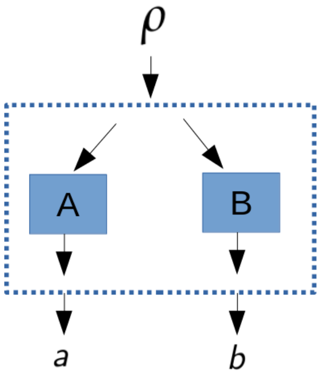
Measurement compatibility

Commutation $\implies$ measurement compatibility

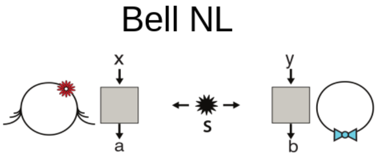


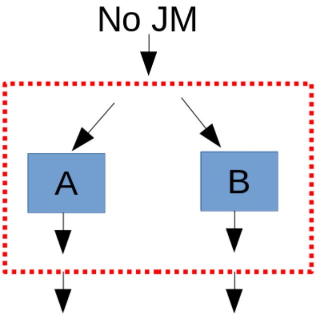
Measurement compatibility

Joint measurability



Bell NL and no JM





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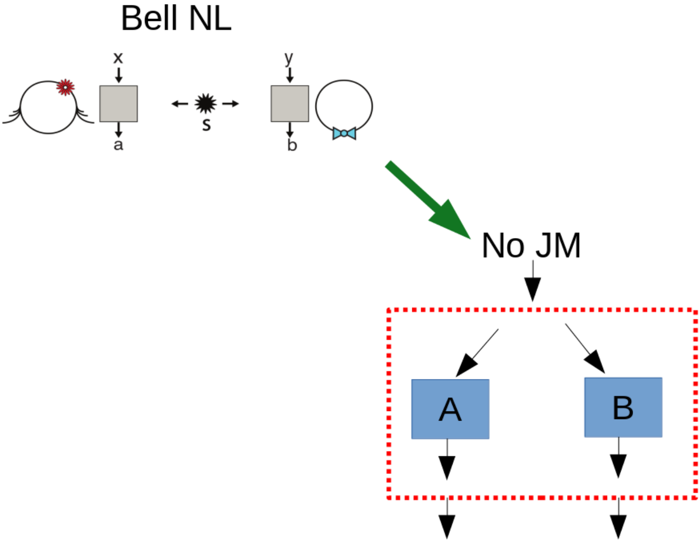
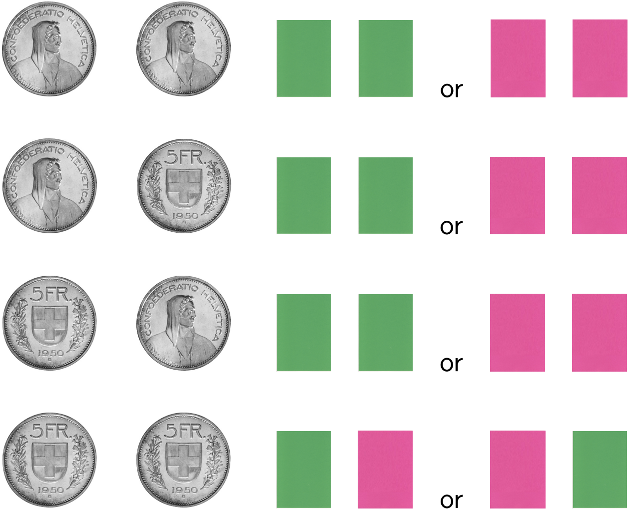


Bell NL

Correlator:

$$\langle A_x B_y \rangle = p(a = b|xy) - p(a \neq b|xy)$$

Winning Conditions



Bell NL

Correlator:

$$\langle CHSH \rangle := \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle$$

Bell NL

The CHSH inequality:

$$\langle CHSH \rangle := \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \overset{\text{local}}{\leq} 2$$

Bell NL

Quantum Observable:

$$A_x := A_{g|x} - A_{p|x}$$
$$B_x := B_{g|x} - B_{p|x}$$

$$\begin{aligned}\langle A_x B_y \rangle_Q &= \mathrm{tr}(\rho A_x \otimes B_y) \\ &= \mathrm{tr}(\rho A_{g|x} \otimes B_{g|y}) + \mathrm{tr}(\rho A_{p|x} \otimes B_{p|y}) \\ &\qquad - \mathrm{tr}(\rho A_{g|x} \otimes B_{p|y}) - \mathrm{tr}(\rho A_{g|x} \otimes B_{g|y})\end{aligned}$$

Bell NL

Quantum Observable:

$$A_x := A_{g|x} - A_{p|x}$$
$$B_x := B_{g|x} - B_{p|x}$$

A_x and B_y are matrices with eigenvalues ± 1

$$\langle A_x B_y \rangle_Q = \mathrm{tr}(\rho A_x \otimes B_y)$$

Bell NL

The CHSH operator:

$$CHSH := A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2$$

Bell NL

The CHSH operator:

$$CHSH := A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2$$

Maximal quantum score is given by the largest eigenvalue of CHSH

Tsirelson Bound

Letters in Mathematical Physics (1980)

QUANTUM GENERALIZATIONS OF BELL'S INEQUALITY

B.S. CIREL'SON
Leningrad, U.S.S.R.

ABSTRACT. Even though quantum correlations violate Bell's inequality, they satisfy weaker inequalities of a similar type. Some particular inequalities of this kind are proved here. The m

$$
\begin{aligned}
A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 &= \frac{1}{\sqrt{2}}\,(A_1^2 + A_2^2 + B_1^2 + B_2^2) - \\
&- \frac{\sqrt{2}-1}{8}\,((\sqrt{2}+1)(A_1 - B_1) + A_2 - B_2)^2 - \\
&- \frac{\sqrt{2}-1}{8}\,((\sqrt{2}+1)(A_1 - B_2) - A_2 - B_1)^2 - \\
&- \frac{\sqrt{2}-1}{8}\,((\sqrt{2}+1)(A_2 - B_1) + A_1 + B_2)^2 - \\
&- \frac{\sqrt{2}-1}{8}\,((\sqrt{2}+1)(A_2 + B_2) - A_1 - B_1)^2 \\
&\le \frac{1}{\sqrt{2}}\,(A_1^2 + A_2^2 + B_1^2 + B_2^2) \le 2\sqrt{2}\cdot \mathbb{1}\,.
\end{aligned}
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Tsirelson Bound

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& - \frac{\sqrt{2}-1}{8}\,((\sqrt{2}+1)(A_1 - B_2) - A_2 - B_1)^2 - \\
& - \frac{\sqrt{2}-1}{8}\,((\sqrt{2}+1)(A_2 - B_1) + A_1 + B_2)^2 - \\
& - \frac{\sqrt{2}-1}{8}\,((\sqrt{2}+1)(A_2 + B_2) - A_1 - B_1)^2 \\
& \leq \frac{1}{\sqrt{2}}\,(A_1^2 + A_2^2 + B_1^2 + B_2^2) \leq 2\sqrt{2}\cdot\mathbb{1}\,.
\end{aligned}
$$

$$\|A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2\| \leq 2\sqrt{2}$$

Tsirelson Bound 2.0

SYMPOSIUM ON THE FOUNDATIONS OF MODERN PHYSICS, pp. 441-460
edited by P. Lahti & P. Mittelstaedt

QUANTUM AND QUASI-CLASSICAL ANALOGS OF BELL INEQUALITIES*

L. A. Khalfin, B. S. Tsirelson

Steklov Mathematical Institute, Leningrad D-11, USSR

Both (1) and (2) can be treated as a consequences of the following inequality:

$$(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2)^2 \leq 4 \cdot \mathbb{1} - [A_1, A_2] \cdot [B_1, B_2], \qquad (3)$$

holding under the same assumptions as (2). From (3) it follows immediately

$$\| A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \| \leq \sqrt{4 + \|[A_1, A_2]\| \cdot \|[B_1, B_2]\|}. \qquad (4)$$

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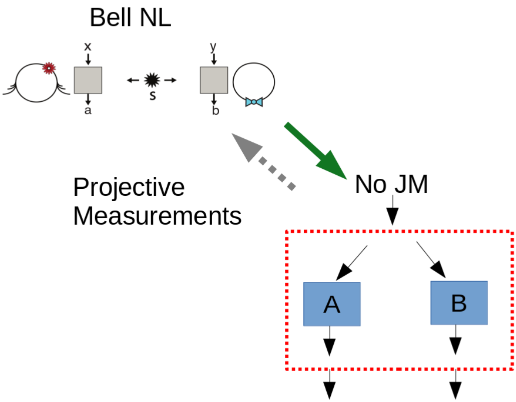
$$\|CHSH\| = \sqrt{4I + \|[A_1,A_2]\|\ \|[B_1,B_2]\|}$$

Tsirelson Bound

$$\|CHSH\| = \sqrt{4I + \|[A_1, A_2]\| \; \|[B_1, B_2]\|}$$

Bell NL and JM



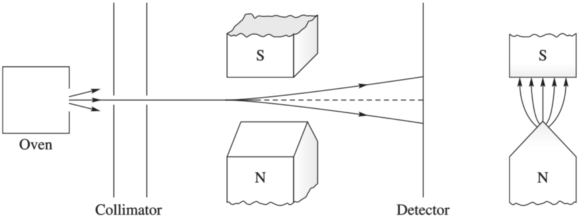
Bell NL and JM

Beyond projective measurements?
Beyond CHSH?

Quantum measurement

Naimark dilation:

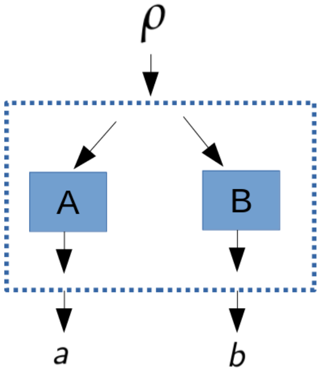
$$\text{POVM} \iff \text{global unitary} + \text{measurement}$$



Joint Measurability

$\{A_a\}$ and $\{B_b\}$ are JM if there exists a measurement $\{M_{ab}\}$ s. t.:

$$\sum_a M_{ab} = B_b$$
$$\sum_b M_{ab} = A_a$$

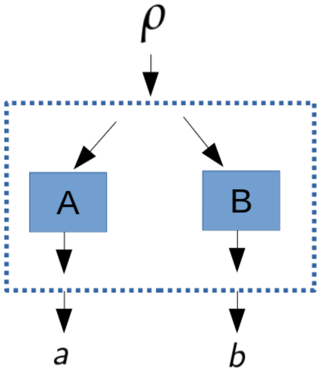


Joint Measurability

$\{A_a\}$ and $\{B_b\}$ are JM if there exists a measurement $\{M_{ab}\}$ s. t.:

$$\sum_a M_{ab} = B_b$$
$$\sum_b M_{ab} = A_a$$

$$M_{ab} \geq 0, \quad \sum_{ab} M_{ab} = I$$



Pauli Measurements

$$\sigma_Z : \{|0\rangle\langle 0|,\ |1\rangle\langle 1|\} \quad \sigma_X : \{|+\rangle\langle +|,\ |-\rangle\langle -|\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2}\ ; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2}\ ; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{X,\eta} : \left\{ \eta|+\rangle\langle +| + (1-\eta)\frac{I}{2}\ ; \qquad \eta|-\rangle\langle -| + (1-\eta)\frac{I}{2} \right\}$$

Noise Pauli Measurements

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2} \,; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{X,\eta} : \left\{ \eta|+\rangle\langle +| + (1-\eta)\frac{I}{2} \,; \qquad \eta|-\rangle\langle -| + (1-\eta)\frac{I}{2} \right\}$$

$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Joint Measurability}$$

Hollow Triangle

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2} \, ; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{X,\eta} : \left\{ \eta|+\rangle\langle +| + (1-\eta)\frac{I}{2} \, ; \qquad \eta|-\rangle\langle -| + (1-\eta)\frac{I}{2} \right\}$$

$$\sigma_{Y,\eta} : \left\{ \eta|Y+\rangle\langle Y+| + (1-\eta)\frac{I}{2} \, ; \qquad \eta|Y-\rangle\langle Y-| + (1-\eta)\frac{I}{2} \right\}$$

$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Pairwise Measurability}$$

Hollow Triangle

$$\sigma_{Z,\eta} : \left\{ \eta|0\rangle\langle 0| + (1-\eta)\frac{I}{2} \; ; \qquad \eta|1\rangle\langle 1| + (1-\eta)\frac{I}{2} \right\}$$

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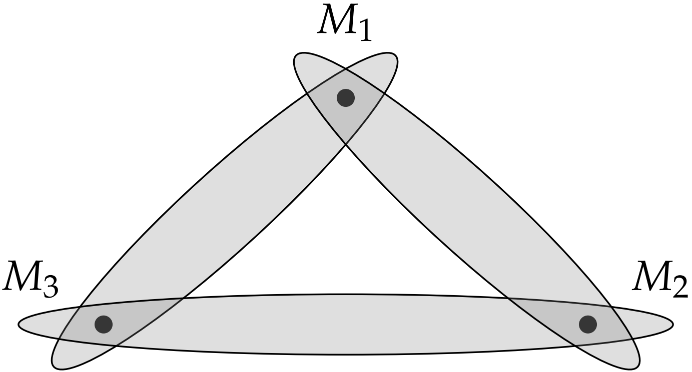
$$\eta \leq \frac{1}{\sqrt{2}} \implies \text{Pairwise Measurability}$$

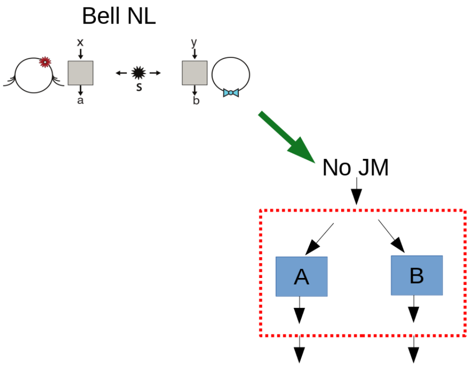
$$\eta \leq \frac{1}{\sqrt{3}} \implies \text{Triplewise Measurability}$$

Hollow Triangle

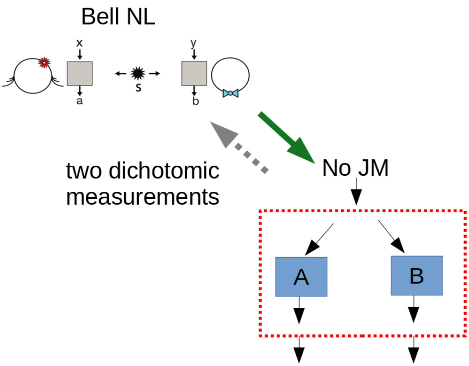


T. Heinosaari, D. Reitzner, P. Stano: Foundations of Physics (2008)

Bell NL and JM



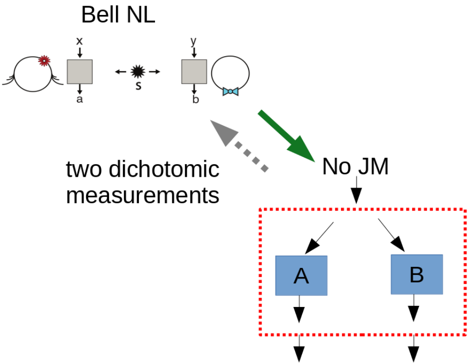
Bell NL and JM



M. Wolf, D. Perez-Garcia, C. Fernandez, PRL (2009)

Bell NL and JM

$\{A_{a|x}\}_{a,x=1}^{2}$ not JM $\implies$ $\exists \rho_{AB}$ and $\{B_{b|y}\}$ such that:
$p(ab|xy) = \mathrm{tr}(\rho_{AB}\, A_{a|x} \otimes B_{b|y})$ is Bell NL



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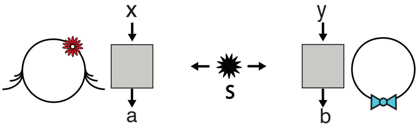
EPR Steering

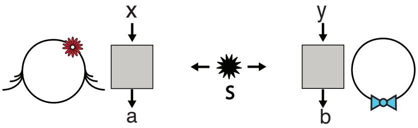
Another manifestation of quantum non-locality

Bell nonlocality

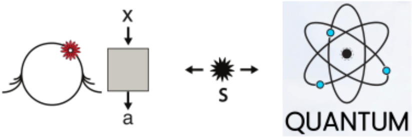


$$p(ab|xy) \neq \sum_{\lambda} \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

Bell nonlocality and EPR Steering

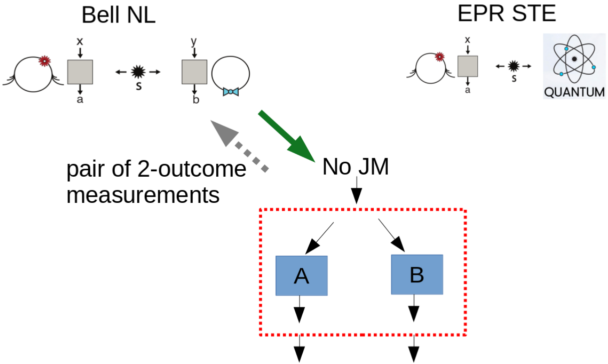


$$p(ab|xy) \neq \sum_\lambda \pi(\lambda) p_A(a|x,\lambda) p_B(b|y,\lambda)$$

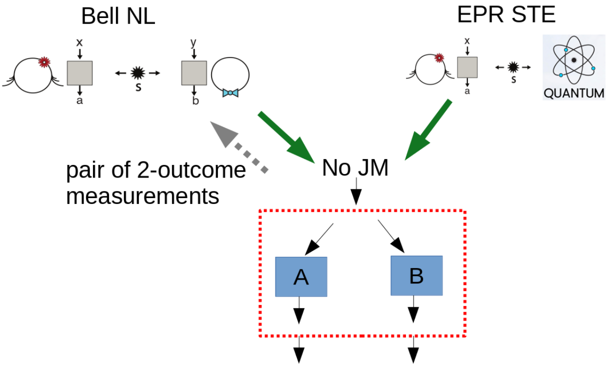


$$p(ab|xy) \neq \sum_\lambda \pi(\lambda) p_A(a|x,\lambda) \mathrm{tr}(\rho_\lambda B_{b|y})$$

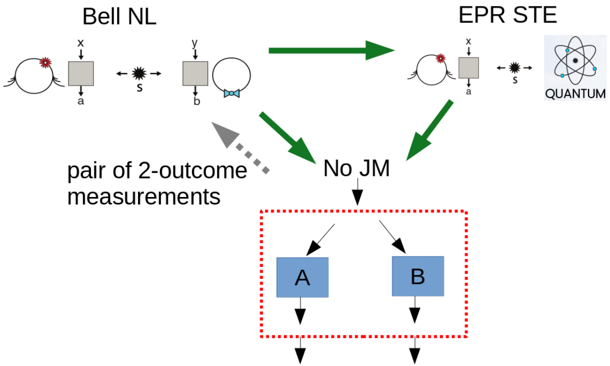
Bell NL, JM, and EPR STE



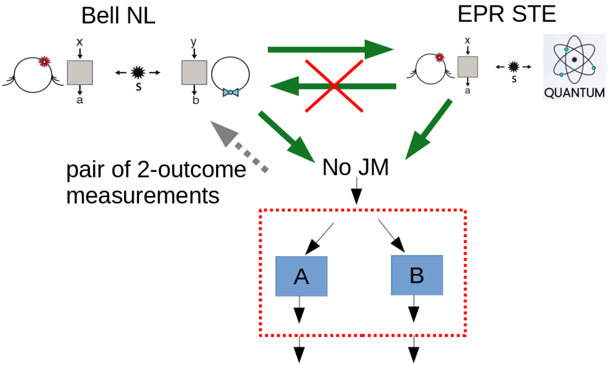
Bell NL, JM, and EPR STE



Bell NL, JM, and EPR STE

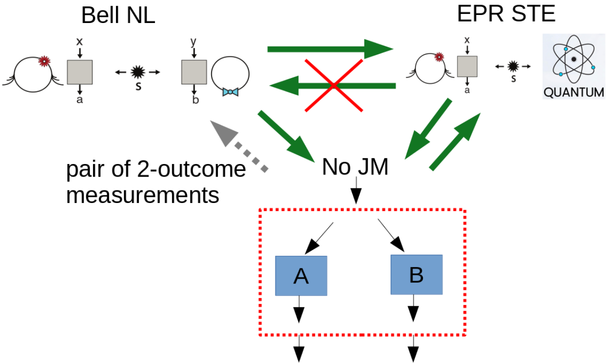


Bell NL, JM, and EPR STE



H. M. Wiseman, S. J. Jones, A. C. Doherty, PRL (2007)
A. Acin, N. Gisin, and .B Toner, PRA (2006)
B. S. Tsirelson, J. Soviet Math. (1987)
M.T. Quintino, T. Vertesi, N. Brunner, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, N. Brunner PRA (2015)

Bell NL, JM, and EPR STE

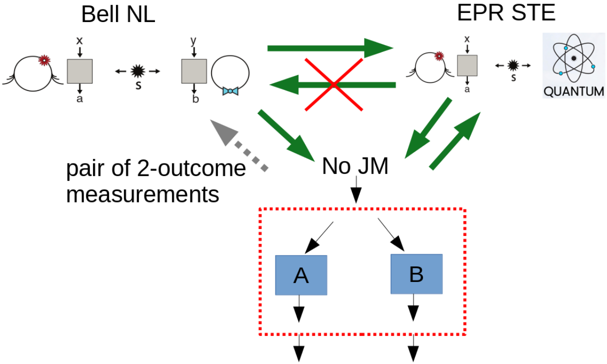


M.T. Quintino, T. Vertesi, N. Brunner PRL (2014)
R. Uola, T. Moroder, and O. Gühne PRL (2014)

Bell NL, JM, and EPR STE

$\{A_{a|x}\}$ not JM
ρ_{AB} is pure and entangled
$\{B_{b|y}\}$ is informationally complete:

$$p(ab|xy) = \mathrm{tr}(\rho_{AB}\, A_{a|x} \otimes B_{b|y}) \text{ is EPR Steerable}$$



M.T. Quintino, T. Vertesi, N. Brunner PRL (2014)
R. Uola, T. Moroder, and O. Gühne PRL (2014)

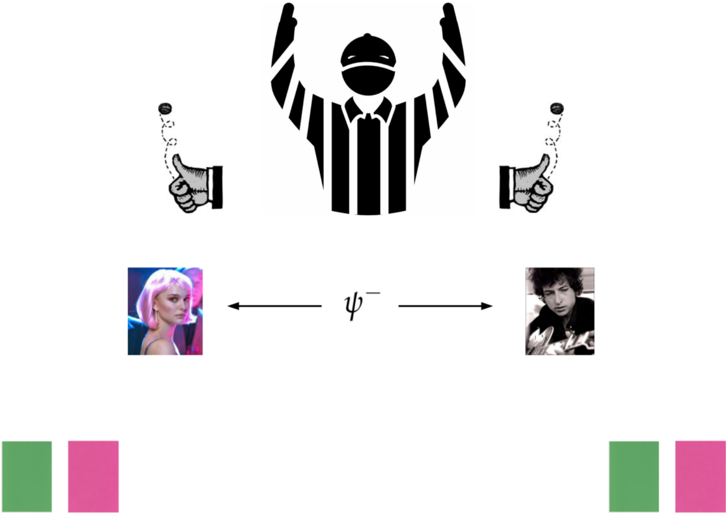
Bell NL, JM, and EPR STE

OK... EPR steering is nice, but, what about Bell?

Bell nonlocality

Local hidden variable model

Quantum Scenarios



Local hidden variable model

$$W_\eta := \eta|\phi^+\rangle\langle\phi^+| + (1-\eta)\frac{I}{4}$$

$$|\phi^+\rangle := \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

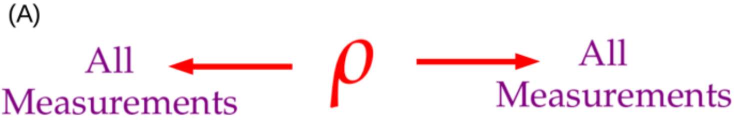
Local hidden variable model

$$W_\eta := \eta|\phi^+\rangle\langle\phi^+| + (1-\eta)\frac{I}{4}$$

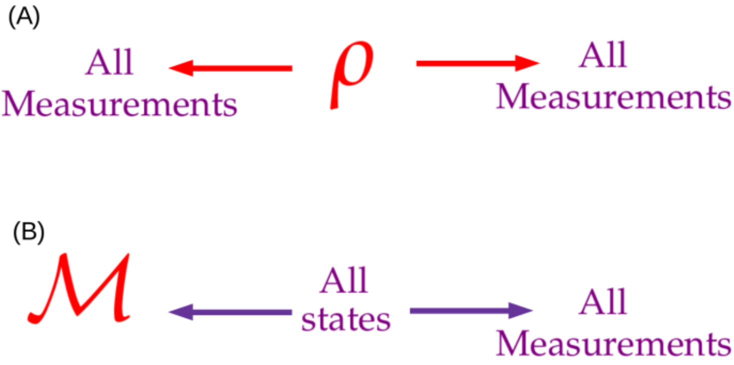
For $\eta \leq 1/2$, there exists a Local Hidden Variable model

R. Werner, PRA (1989)

Local hidden variable model



Local hidden variable model



Bell nonlocality

$$p(ab|xy) = \mathrm{tr}(\rho_{AB}^{\eta} A_{a|x} \otimes B_{b|y})$$

$$\rho_{AB}^{\eta} := \eta\rho_{AB} + (1-\eta)\frac{I}{d} \otimes \rho_B$$

$$\rho_B := \mathrm{tr}_A(\rho_{AB})$$

Bell nonlocality

$$\mathrm{tr}(\rho^{\eta}_{AB}\, A_{a|x} \otimes B_{b|y}) = \mathrm{tr}(\rho_{AB}\, A^{\eta}_{a|x} \otimes B_{b|y})$$

$$A^{\eta}_{a|x} := \eta A_{a|x} + (1-\eta)\frac{I}{d}\,\mathrm{tr}(A_{a|x})$$

Local hidden variable model

LHV model for the class:

$$W_{\eta,\theta} := \eta|\phi^\theta\rangle\langle\phi^\theta| + (1-\eta)\frac{I}{2} \otimes \rho_\theta$$

$$|\phi^\theta\rangle := \sin(\theta)|00\rangle + \cos(\theta)|11\rangle, \quad \rho_\theta := \mathrm{tr}_A(|\phi^\theta\rangle\langle\phi^\theta|$$

in a range where there are noisy incompatible measurements

Local hidden variable model

$$W_{\eta,\theta} := \eta|\phi^\theta\rangle\langle\phi^\theta| + (1-\eta)\frac{I}{2}\otimes\rho_\theta$$

$$\cos^2(\theta) \geq \frac{2\eta - 1}{(2-\eta)\eta^3} \implies \text{LHV model}$$

J. Bowles, F. Hirsch, M.T. Quintino, N. Brunner, PRL (2016)

Local hidden variable model

$$W_{\eta,\theta} := \eta|\phi^{\theta}\rangle\langle\phi^{\theta}| + (1-\eta)\frac{I}{2}\otimes\rho_{\theta}$$

$$\cos^2(\theta) \geq \frac{2\eta - 1}{(2-\eta)\eta^3} \implies \text{LHV model}$$

For $\eta > 1/2$, $\{A_{a|x}\}$ is no JM.
But for $\theta = \pi/4$ the model returns $\eta = 1/2\ldots$

Local hidden variable model

For the maximally entangled state, we have Grothendieck!

Local hidden variable model

Journal of Soviet Mathematics,
vol. 36, number 4, pp. 557–570.
February, 1987

QUANTUM ANALOGUES OF THE BELL INEQUALITIES. THE CASE OF TWO SPATIALLY SEPARATED DOMAINS

B. S. Tsirel'son

UDC 519.2

significantly simpler manner in terms of vectors in a Euclidean space. Then it becomes clear that the constant K. is nothing else but Grothendieck's well known constant K_G, investigated by mathematicians from 1956 up to now!

Regarding the problem of the Grothendieck constant, we refer the reader to [11]. It is proved there that

$$K_G \lesssim \frac{\pi}{2 \ln(1+\sqrt{2})} \approx 1.782.$$

Local hidden variable model

$$W_\eta := \eta|\phi^+\rangle\langle\phi^+| + (1-\eta)\frac{I}{4}$$

LHV for $\eta \leq \frac{1}{K_G(3)}$, and $K_G(3) < 2$

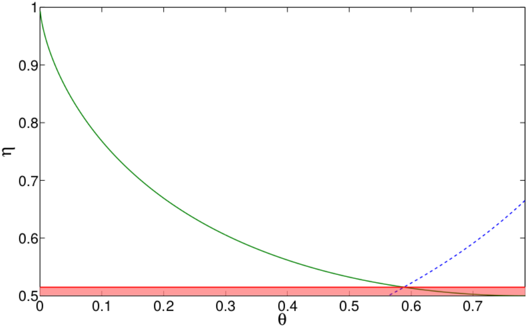
A. Acin, N. Gisin, B. Toner, PRA (2006)
B. Tsirelson, J. Soviet Math. (1987)
J. L. Krivine, Adv. Math. (1979)

Local hidden variable model

This Grothendieck approach only work for projective measurements...

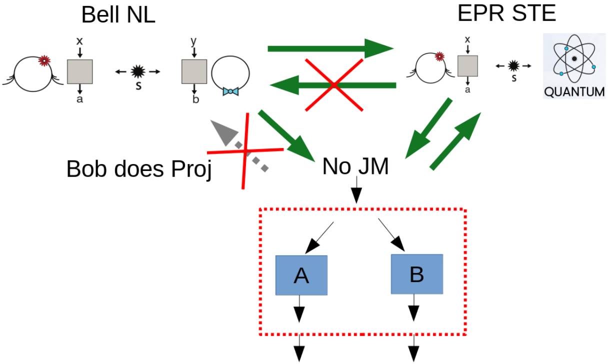
Bell NL and JM

LHV Model+Grothendieck+LHV extention based on PPT:



M.T. Quintino, J. Bowles, F. Hirsch, N. Brunner, PRA (2016)

Bell NL, JM, and EPR STE



Bell NL, JM, and EPR STE

Nice, but. . .

Bob can do POVMs. . .

Bell NL, JM, and EPR STE

Nice, but. . .

Bob can do POVMs. . .

Normally, constructing LHV models is requires a lot of creativity

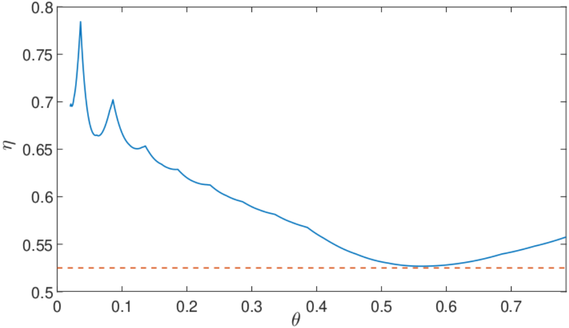
Local hidden variable model

How about using the computer to find LHV models for us?

M.T. Quintino, F. Hirsch, T. Vertesi, M. Pusey, N. Brunner, PRL (2016)
D. Cavalcanti, L. Guerini, R. Rabelo, P. Skrzypczyk, PRL (2016)

Bell NL and JM

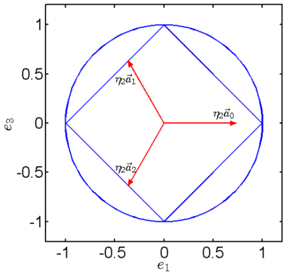
Challenge accepted:

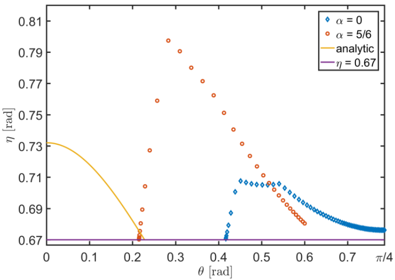


F. Hirsch, M.T. Quintino, N. Brunner, PRA (2018)

Bell NL and JM

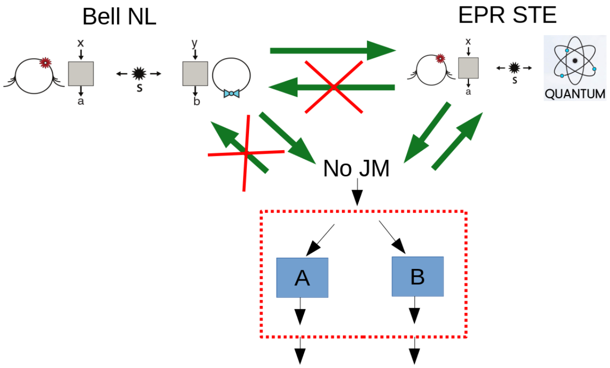
Three planar measurements:





E. Bene, T. Vertesi, NJP (2018)

Bell NL, JM, and EPR STE



Open questions

- Direct/intuitive LHV model for incompatible measurements?

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- Direct/intuitive LHV model for incompatible measurements?
- Simple criteria for measurement Bell NL?
- Natural manners to "activate" measurement NL?
- How measurement locality relate to other areas of quantum info?

Thank you!

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