Quantum information processing via higher-order operations

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This thesis is dedicated to all the students I've had the opportunity to work with—undergraduate, master's, and PhD—past, present, and future.

Abstract

This thesis investigates quantum information processing within the framework of higher-order operations, focusing on the transformation and discrimination of quantum channels, as well as on the role of indefinite causal order in enhancing quantum tasks. Following a preliminary chapter on the fundamentals of higher-order quantum operations, we present two chapters of original research.

The first research chapter addresses the transformation of unitary quantum operations, examining the problem of converting multiple calls of an arbitrary unitary operation into its inverse, transpose, and complex conjugate. We analyse the performance of parallel, sequential, and indefinite-causal-order strategies, revealing insights into the conditions under which sequential strategies can be parallelised, and conditions where indefinite-causal-order strategies are a useful resource. There we stablish one-to-one relationships with parallel unitary inversion problems, to unitary estimation problems, and the seemingly unrelated task of port-based teleportation.

The second research chapter explores quantum channel discrimination when multiple calls are available. As with the unitary transformation analysis, we evaluate parallel, sequential, and indefinite-causal-order strategies. Traditionally, the field of quantum channel discrimination has focused on sets of unitary channels that form a group and are uniformly distributed, or discrimination tasks involving only two channels. Here, we combine a computer-assisted proof method with semidefinite programming techniques, arising from the framework of higher-order quantum operations, to broaden the scope of problems that are tractable. Using our methods, we identify and present various examples of ensembles of channels where sequential strategies offer an advantage when compared to parallel ones.

Finally, we provide an outlook on these different research directions and list recent contributions to other fields.

Acknowledgements

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Thesis structure

In this thesis, we reserve the theorem environment exclusively for results originally presented in a publication of the author. This can be contrasted to the proposition environment, which presents results proven in articles not co-written by the author.

In Chapter 1, we present a brief introduction to quantum information and higher-order quantum operations.

In Chapter 2, we review the required mathematical definitions for quantum information and present the basic concepts for higher-order quantum operations.

In Chapter 3, we present original results from articles co-written by the author, specifically focusing on the problem of transforming quantum unitary channels. These works explore whether multiple calls of a quantum operation can be combined to perform transformations such as inversion, transposition, and complex conjugation, comparing the effectiveness of parallel, sequential, and indefinite-causal-order strategies.

In Chapter 4, we present original results from articles co-written by the author, focusing on the discrimination of quantum channels. This chapter addresses methods for distinguishing among different quantum channels, employing techniques like semidefinite programming and computer-assisted proofs to tackle various discrimination tasks beyond traditional symmetric cases.

In Chapter 5, we list the articles from the author published after their PhD, but that were not included in the earlier chapters.

In Chapter 6, we summarise and reflect on the main findings of this thesis, discuss open questions and possible future directions of the field.

Contents

Chapter 1

Introduction

Quantum information is a branch of science that approaches quantum mechanics from an information theory perspective. In this approach, key concepts in standard information theory such as bits, channels, circuits, and input/output relations are extended into the formalism of quantum theory. This information view of quantum mechanics provides important basis for quantum technologies and have already led to breakthroughs related to quantum foundations, quantum computation, and quantum communication. Among others, celebrated results of quantum information include the impossibility of cloning general quantum states [1], a protocol for teleporting quantum states [2], compressing communication via quantum superdense coding [3], quantum-based cryptography [4], and device-independent protocols based on Bell nonlocality [5].

A *quantum circuit* is a model for quantum computation consisting of a sequence of quantum gates and quantum measurements. The decomposition of quantum circuits into elementary gates establishes the basis for analysing complexity in *quantum computing* and provides a useful framework to understand quantum information protocols. A typical example in terms of applications is the celebrated Shor algorithm, a quantum-circuit that can be used to factorise integers efficiently [6].

Quantum operations form an important pillar of quantum theory and a key point for many applications to quantum technologies. Traditionally, quantum operations were only viewed as devices to transform quantum states, such as quantum communication channels between distant parties or quantum gate elements in a quantum circuit. However, quantum operations themselves can also be subjected to transformations and, in this way, play the role of a state by a *higher-order quantum operation* (HOQO). A simple example of a HOQO is a quantum circuit with missing gates, as illustrated in Fig. 1.1.

The HOQO approach has given rise to powerful mathematical methods to analyse quantum circuits, and problems involving quantum operations and quantum measurements. In particular, it has allowed several such problems to be formalised as semidefinite programs (SDP) and the symmetries that often appear in these problems to be analysed and treated with group-representation theory methods. Causally ordered HOQO appeared in the literature under the name of *quantum combs* [7, 8], *quantum strategies* [9], and *quantum channels with memory* [10], and have led to important results in tasks such as quantum channel discrimination [11–13], quantum metrology [14–16], tomography on quantum

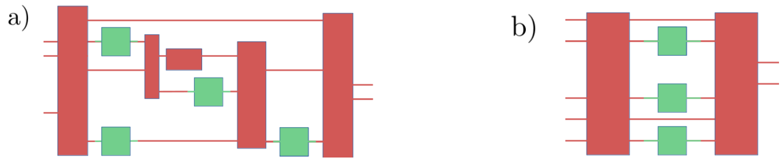


Figure 1.1: Pictorial representation of a quantum circuit. Elements in red represent a (casually ordered) HOQO which transforms input operations. Elements in green are the input operations, which can be plugged into and unplugged from the circuit. Panel a) depicts a general circuit, while panel b) depicts a parallel circuit, where the input operations can be used simultaneously.

processes [17], controlling dynamics of quantum systems [18–20], universal transformation of unitary gates [13, 21], quantum causal-effect analysis [22] and optimal methods to store the action of quantum operations into a quantum memory state [23–26].

Additionally, differently from states, operations have a clear notion of input and output. Hence, when considering transformations between two or more operations, the concept of causal order naturally emerges. Interestingly, the postulates of quantum mechanics do not explicitly forbid the existence of HOQO which do not respect any definite causal order between the uses of the input operations. This leaves room for the existence of quantum process with indefinite causality [27, 28] and for a fruitful analysis on how causality can be understood in quantum theory.

Chapter 2

Preliminaries

2.1 Linear algebra using the bra-ket notation and the Choi representation

2.1.1 Vectors

As is standard in quantum information, we denote inner-product vector spaces by $\mathcal{H}$, as a reference to Hilbert spaces. All vector spaces considered in this thesis are finite-dimensional linear vector spaces over the field of complex numbers, that is, isomorphic to $\mathbb{C}^d$ for some non-zero natural number $d \in \mathbb{N}$. Additionally, since this work only considers finite-dimensional linear spaces, the metric space induced by the inner product is necessarily complete, meaning all inner-product linear spaces discussed here are trivially Hilbert spaces.

In bra-ket notation, also referred to as Dirac notation, vectors from $\mathcal{H}$ are represented by "kets", e.g, $|\psi\rangle \in \mathcal{H}$, and linear functionals from the dual space are represented by "bras", e.g., $\langle\phi| \in \text{dual}(\mathcal{H}) \cong \mathcal{H}$. The functional$\langle\phi| \in \text{dual}(\mathcal{H})$ is related to the vector $|\phi\rangle \in \mathcal{H}$ via Riesz representation lemma. That is, when "a bra meets a ket" we have $\langle\phi|\psi\rangle = \big\langle\, |\phi\rangle\, , |\psi\rangle\, \big\rangle$, with $\big\langle\, \cdot\, , \cdot \big\rangle$ denotes the standard inner product in $\mathbb{C}^d$, which is linear in the second component and anti-linear in the first.

The set of vectors $\{|i\rangle\}_{i=0}^{d-1} \subseteq \mathbb{C}^d$ is called the computational basis and forms an orthonormal basis for $\mathbb{C}^d$, that is $\langle i|j\rangle = \delta_{ij}$. An arbitrary vector $|\psi\rangle \in \mathbb{C}^d$ can be decomposed as $|\psi\rangle = \sum_i \gamma_i |i\rangle$, where $\gamma_i = \langle i|\psi\rangle \in \mathbb{C}$. We use the star symbol $*$ to denote complex conjugation of complex numbers, that is, γ_i^* is the complex conjugation of $\gamma_i \in \mathbb{C}$. In this way, complex conjugate of a vector is defined via $|\psi\rangle^* := \sum_i \gamma_i^* |i\rangle$, and its transpose —denoted by the symbol ^{T}— via $|\psi\rangle^T := \langle\psi|^* = \sum_i \gamma_i \langle i|$. This allows us to express the dual vector corresponding to $|\psi\rangle$ as $|\psi\rangle^\dagger := \langle\psi| = |\psi\rangle^{*T}$.

2.1.2 Linear operators

In bra-ket notation, an arbitrary linear operator $A : \mathcal{H}_\text{I} \to \mathcal{H}_\text{O}$ can be written as $A = \sum_{ij} \gamma_{ij} |i\rangle\langle j|$ where $\gamma_{ij} \in \mathbb{C}$ are complex coefficients that respect $\gamma_{ij} = \langle i| A |j\rangle$. As with vectors, the complex conjugate and transposition of linear operators are defined with respect to the computational basis, that is,

$$A^* := \sum_{ij} \gamma_{ij}^* |i\rangle\langle j| \tag{2.1}$$

$$A^T := \sum_{ij} \gamma_{ij} \Big(|i\rangle\langle j| \Big)^T = \sum_{ij} \gamma_{ij} |j\rangle\langle i| . \tag{2.2}$$

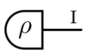
With these definitions, the adjoint[1] of a linear operator A can be written as $A^\dagger = A^{*T} = \sum_{ij} \gamma_{ij}^* |j\rangle\langle i|$. An operator A is self-adjoint if $A = A^\dagger$.

The set of all operators mapping some space $\mathcal{H}$ to itself is denoted by $\mathcal{L}(\mathcal{H})$, that is, writing $T \in \mathcal{L}(\mathcal{H})$ is a compact notation for $T : \mathcal{H} \to \mathcal{H}$. Also, notice that since our linear spaces are finite-dimensional, all linear operators are trivially bounded.

The trace of a linear operator $A \in \mathcal{L}(\mathcal{H})$ is defined as $\operatorname{tr}(A) := \sum_i \langle i| A |i\rangle = \sum_i \gamma_{ii}$. A linear operator $\rho \in \mathcal{L}(\mathcal{H})$ is positive semidefinite if $\langle\psi| \rho |\psi\rangle \geq 0$ for every vector $|\psi\rangle \in \mathbb{C}^d$. We write $\rho \geq 0$ to denote that ρ is positive semidefinite, and we write $A \geq B$ to indicate $A - B \geq 0$. When $\mathcal{H}$ is a finite-dimensional complex linear space (the case considered in this thesis), an operator $A \in \mathcal{L}(\mathcal{H})$ is positive semidefinite if and only if (i) $A = A^\dagger$ and (ii) the eigenvalues of A are non-negative real numbers.

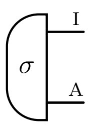
When analysing higher-order transformations, it is useful to represent linear operators and linear maps in the (quantum) circuit notation. This notation is often intuitive and sometimes self-explanatory, and will be gradually introduced in this thesis. The advantage of this notation is that it allows for better visualisation of mathematical equations and, in particular, clearly indicates the domain and codomain of mathematical objects. We emphasise that the circuit notation is more than just a "drawing" to illustrate concepts; it is instead a rigorous mathematical description for equations involving linear operators and linear maps.

In circuit notation[2], an arbitrary linear operator $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I})$ is represented as



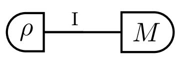
$$:= \rho \in \mathcal{L}(\mathcal{H}_\mathrm{I}). \tag{2.3}$$

Also, a bipartite linear operator $\sigma \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{A})$ is represented as



$$:= \sigma \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{A}). \tag{2.4}$$

In circuit notation, we also define the trace of an arbitrary operator $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I})$ with another arbitrary operator $M \in \mathcal{L}(\mathcal{H}_\mathrm{I})$ via



$$:= \operatorname{tr}(\rho M) \in \mathbb{C}. \tag{2.5}$$

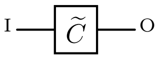
[1]The adjoint of a linear operator $A : \mathcal{H}_\mathrm{I} \to \mathcal{H}_\mathrm{O}$ is the unique operator $A^\dagger : \mathcal{H}_\mathrm{O} \to \mathcal{H}_\mathrm{I}$ such that $\big\langle |\phi\rangle , A|\psi\rangle \big\rangle = \big\langle A^\dagger |\phi\rangle , |\psi\rangle \big\rangle$ for every $|\psi\rangle \in \mathcal{H}_\mathrm{I}$ and every $|\phi\rangle \in \mathcal{H}_\mathrm{O}$.

[2]This work uses Quantikz [29] to draw quantum circuits with latex.

2.1.3 Linear maps

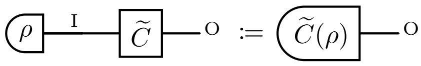
In this work, and in some branches of quantum information, we reserve the terminology *linear map* for transformations between linear operators. For instance, a linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_\text{I}) \to \mathcal{L}(\mathcal{H}_\text{O})$ is a linear transformation from linear operators acting on $\mathcal{H}_\text{I}$ to operators acting on $\mathcal{H}_\text{O}$, where I and O stands for input and output, respectively. Here, we adopt the convention of denoting linear maps with a tilde, so the role of each mathematical object can be easily identified.

In circuit notation, a linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_\text{I}) \to \mathcal{L}(\mathcal{H}_\text{O})$ is represented as



$$:= \widetilde{C} : \mathcal{L}(\mathcal{H}_\text{I}) \to \mathcal{L}(\mathcal{H}_\text{O}). \tag{2.6}$$

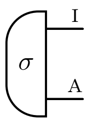
Also, the application of an arbitrary map $\widetilde{C}$ onto an operator $\rho \in \mathcal{L}(\mathcal{H}_\text{I})$ is denoted as



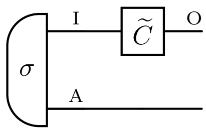
(2.7)

$$= \widetilde{C}(\rho) \in \mathcal{L}(\mathcal{H}_\text{O}). \tag{2.8}$$

A linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_\text{I}) \to \mathcal{L}(\mathcal{H}_\text{O})$ is trace-preserving (TP) if, for any operator $\rho \in \mathcal{L}(\mathcal{H}_\text{I})$, we have that $\text{tr}\Big(\widetilde{C}(\rho)\Big) = \text{tr}(\rho)$. A linear map $\widetilde{C}$ is positive if for every $\rho \geq 0$ we have that $\widetilde{C}(\rho) \geq 0$. A linear map is completely positive (CP) if, for any linear space $\mathcal{H}_\text{A}$, the map $\widetilde{C} \otimes \widetilde{\mathbb{1}}_\text{A} : \mathcal{L}(\mathcal{H}_\text{I} \otimes \mathcal{H}_\text{A}) \to \mathcal{L}(\mathcal{H}_\text{O} \otimes \mathcal{H}_\text{A})$ is positive, where $\widetilde{\mathbb{1}}_\text{A} : \mathcal{L}(\mathcal{H}_\text{A}) \to \mathcal{L}(\mathcal{H}_\text{A})$ stands for the identity map, i.e., $\widetilde{\mathbb{1}}_\text{A}(A) = A$ for any $A \in \mathcal{L}(\mathcal{H}_\text{A})$. In other words, a map $\widetilde{C}$ is CP if, for any positive semidefinite linear operator $\sigma \geq 0$ in $\sigma \in \mathcal{L}(\mathcal{H}_\text{I} \otimes \mathcal{H}_\text{A})$, it holds that $[\widetilde{C} \otimes \widetilde{\mathbb{1}}_\text{A}](\sigma) \geq 0$. In circuit notation, a linear map $\widetilde{C}$ is CP if for every $\sigma \in \mathcal{L}(\mathcal{H}_\text{I} \otimes \mathcal{H}_\text{A})$ we have the implication



$\geq 0 \implies$



$\geq 0.$ (2.9)

2.1.4 The Choi representation of operators and maps

It is not hard to check that linear operators satisfy the axioms of vectors spaces; hence, linear operators can also be treated as vectors acting on a larger linear space. Analogously, linear maps can be treated as linear operators acting on a larger space. There are various ways to establish a one-to-one correspondence between vectors and linear operators, and between linear maps and linear operators. In quantum information, this one-to-one correspondence is commonly established using the de Pillis-Choi-Jamiołkowski isomorphism [30–32], here simply referred to as the Choi representation. This isomorphism is defined via a linear invertible function that allows us to write linear operators as vectors, and linear maps as linear operators.

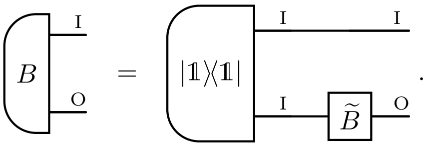
Definition 1 (Choi vector and Choi operator)**.** *Let $A : \mathcal{H}_\mathrm{I} \to \mathcal{H}_\mathrm{O}$ be a linear operator. The Choi vector of A is a vector*[3] $|A\rangle \in \mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O}$ *defined via*

$$|A\rangle := \sum_i |i\rangle \otimes \Big(A\,|i\rangle\Big) \tag{2.10}$$

Let $\widetilde{B} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ be a linear map. The Choi operator of $\widetilde{B}$ is a linear operator $B \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O})$ defined via

$$B := \sum_{ij} |i\rangle\langle j| \otimes \widetilde{B}(|i\rangle\langle j|). \tag{2.11}$$

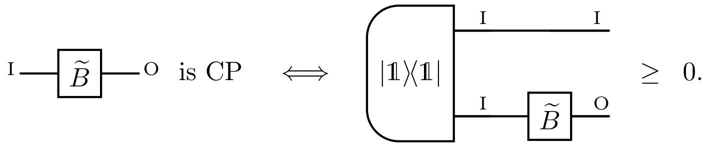
The Choi vector of the identity operator $\mathbb{1} \in \mathcal{L}(\mathbb{C}^d)$ is $|\mathbb{1}\rangle = \sum_i |i\rangle \otimes |i\rangle$, and it is proportional to the maximally entangled state $\left|\phi_d^+\right\rangle := \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i\rangle$. The Choi vector of any linear operator A can be written as $|A\rangle = [\mathbb{1} \otimes A]\,|\mathbb{1}\rangle$. Also, for any linear map $\widetilde{B}$, its Choi operator can be written as $B = \left[\widetilde{\mathbb{1}} \otimes \widetilde{B}\right](|\mathbb{1}\rangle\langle\mathbb{1}|)$. Thus, in circuit notation, the Choi operator B of a linear map $\widetilde{B}$ can be defined as



(2.12)

We also notice that linear maps $\widetilde{C} : \mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ that can be written as $\widetilde{C}(\rho) = U\rho U^\dagger$ for some linear operator $U : \mathcal{H}_I \to \mathcal{H}_O$ have their Choi operator given by $C = |U\rangle\langle U|$.

In a seminal paper [32], Choi proved that a linear map $\widetilde{C}$ is CP if and only if its Choi operator C is positive semidefinite, that is

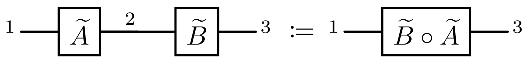


(2.13)

Since quantum channels and quantum instruments are necessarily CP maps, the Choi representation is a convenient tool for representing linear maps in quantum theory.

2.1.5 Composing maps with the link product

Let $\widetilde{A} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ and $\widetilde{B} : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3)$ be linear maps. Since the output space of $\widetilde{A}$ matches the input space of $\widetilde{B}$, we can define the composition $\widetilde{B} \circ \widetilde{A} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_3)$. In circuit notation, this reads as



(2.14)

$$= \widetilde{B} \circ \widetilde{A} \tag{2.15}$$

[3]Note that is also very common in the literature to use a "double-ket" notation to denote the Choi vectors of linear operations [8, 33, 34]. In the double-ket notation one would write $|A\rangle\rangle$ instead of $|A\rangle$.

Note that the composition of the maps $\widetilde{A}$ and $\widetilde{B}$ appears reversed in circuit notation; this occurs because in circuit notation, "time" flows from left to right, whereas in standard function composition, the order is interpreted right-to-left. It would maybe make sense to have circuits flowing from right to left, but since the left-to-right circuit convention is a well established convention, we decided to follow it.

When dealing with Choi operators, composition of linear maps can be conveniently expressed in terms of the *link product* [7], an operation between linear operators denoted by the star symbol $*$. Before presenting the formal definition of the link product, let us examine the equations:

$$\begin{aligned} &\textbf{Map: } \widetilde{A} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2) && \textbf{Choi: } A \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \\ &\textbf{Map: } \widetilde{B} : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3) && \textbf{Choi: } B \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_3) \\ &\textbf{Map: } \widetilde{B} \circ \widetilde{A} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_3) && \textbf{Choi: } B * A \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_3). \end{aligned} \tag{2.16}$$

The link product is defined such that $B*A$ is the Choi operator of the composed linear map $\widetilde{B} \circ \widetilde{A}$, that is, $\widetilde{B} \circ \widetilde{A} = \widetilde{B * A}$. As shown in Ref. [7], this requirement uniquely defines the operation

$$B * A \coloneqq \mathrm{tr}_2 \left([\mathbb{1}_1 \otimes B] \left[A^{T_2} \otimes \mathbb{1}_3 \right] \right), \tag{2.17}$$

with $(\cdot)^{T_2}$ being the partial transposition on the linear space $\mathcal{H}_2$.

It is also useful to consider the link product between operators that are not defined on a bipartition. For instance, if $B \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_3)$ is the Choi operator of some map $\widetilde{B} : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3)$ and $\rho \in \mathcal{L}(\mathcal{H}_2)$ is a linear operator, we have that[4]

$$B_{23} * \rho_2 \coloneqq \mathrm{tr}_2 \left(B_{23} \left[\rho_2^T \otimes \mathbb{1}_3 \right] \right) \tag{2.18}$$

$$= \widetilde{B}(\rho). \tag{2.19}$$

One way to understand Eq. (2.18) is to see view ρ_2 as an operator acting in $\mathcal{H}_1 \otimes \mathcal{H}_2$, but $\mathcal{H}_1$ is one dimensional.

Using analogous reasoning, if $\rho, M \in \mathcal{L}(\mathcal{H}_1)$, we have that $M_1 * \rho_1 \coloneqq \mathrm{tr}(\rho^T M)$. Also, if $\rho \in \mathcal{L}(\mathcal{H}_1)$ and $\sigma \in \mathcal{L}(H_3)$, $\rho_1 * \sigma_3 \coloneqq (\mathbb{1}_1 \otimes \sigma_3)(\rho_1 \otimes \mathbb{1}_3) = \rho_1 \otimes \sigma_3$. It is also convenient to define the link product between maps, instead of two operators. That is, for any $\widetilde{A} : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$, $\widetilde{B} : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3)$, and $\rho \in \mathcal{L}(\mathcal{H}_1)$, we define

$$\widetilde{B} * \widetilde{A} \coloneqq \widetilde{B * A} \tag{2.20}$$

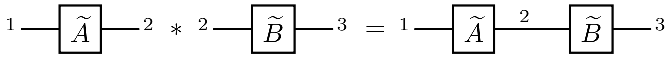
and $\widetilde{A} * \rho \coloneqq A * \rho$.

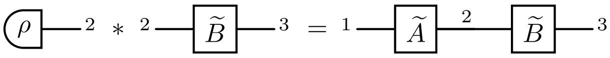
Additionally, since the trace is cyclic, i.e., $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, if we keep track of the linear spaces where the operators act and the domain and codomain of linear maps, the link product is commutative[5] and associative. That is,

[4] In some equations, especially when dealing with the link product, we use subscripts to indicate the spaces where the operators act. Even if sometimes redundant, as when we write $B_{23} * \rho_3$ instead of $B * \rho$, the subscripts can help us identify the linear spaces where the mathematical objects are defined more easily.

[5] It is commutative up to a permutation on the linear spaces where the operators act.

if $B_{23} * A_{12} = C_{13}$, we have that $A_{12} * B_{23} = C_{31}$, and $(C_{34} * B_{23}) * A_{12} = C_{34} * (B_{23} * A_{12})$. In circuit notation, the link product reads as following:


(2.21)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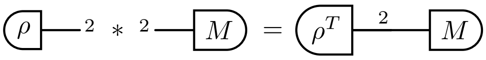
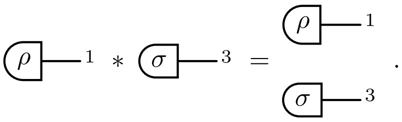
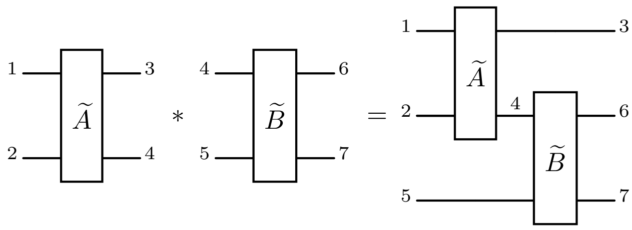
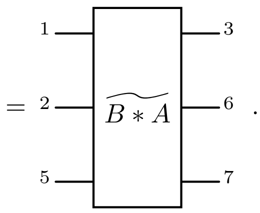

(2.23)


(2.24)

Also, if $\widetilde{A} : \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_3 \otimes \mathcal{H}_4)$ and $\widetilde{B} : \mathcal{L}(\mathcal{H}_4 \otimes \mathcal{H}_5) \to \mathcal{L}(\mathcal{H}_6 \otimes \mathcal{H}_7)$ are linear maps with a multipartite structure, we have that


(2.25)


(2.26)

In a circuit-diagrammatic picture, we can see the link product as: "If the circuit lines correspond to the same linear spaces, we connect them. If the circuit lines correspond to different linear spaces, they should be viewed as parallel, and we do not connect them." Also, as shown in Eq. (2.25), the link product notation is particularly convenient when composing linear maps whose domain and/or codomain have a multipartite structure. Also, as we will see later, the link product is also useful to describe "quantum circuits with open slots".

2.2 Quantum states, channels, measurements, and instruments

Let us now define a quanutm state.

Definition 2 (Quantum state)**.** *A linear operator $\rho \in \mathcal{L}(\mathcal{H})$ is a quantum state if $\rho \geq 0$ and* $\mathrm{tr}(\rho) = 1$.

Deterministic transformations between quantum states are mathematically described by quantum channels, which we define as follows:

Definition 3 (Quantum channel)**.** *A linear map* $\widetilde{C} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ *is a quantum channel if* $\widetilde{C}$ *is completely positive and trace preserving (CPTP).*

Quantum channels are then linear maps that transform quantum states into other quantum states. Also, due to the CP property, they transform quantum states into quantum states even when applied on part of a bipartite state[6].

A linear operator $U : \mathcal{H}_\mathrm{I} \to \mathcal{H}_\mathrm{O}$ is unitary if[7] $U^\dagger U = UU^\dagger = \mathbb{1}$, where $\mathbb{1}$ is the identity operator. A quantum channel $\widetilde{U} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ is unitary if there exists a unitary operator $U : \mathcal{H}_\mathrm{I} \to \mathcal{H}_\mathrm{O}$ such that $\widetilde{U}(\rho) = U\rho U^\dagger$ holds for every linear operator $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I})$, Note that for any $\theta \in \mathbb{R}$, the unitary operators $e^{i\theta}U$ represent the same unitary channel. That is,

$$\widetilde{U}(\rho) = e^{i\theta}U\rho \left(e^{i\theta}U\right)^\dagger = e^{i\theta}e^{-i\theta}U\rho U^\dagger = U\rho U^\dagger. \tag{2.27}$$

In quantum theory, it is common to assume that the unitary operators of unitary channels have determinant one, since this can be done without loss in generality. More precisely, for any linear operator $A \in \mathcal{L}(\mathbb{C}^d)$, and any complex number $\alpha \in \mathbb{C}$, it holds that $\det(\alpha A) = \alpha^d \det(A)$. For this reason, any unitary channel $\widetilde{U} = U\rho U^\dagger$ can be written as $\widetilde{U} = U'\rho U'^\dagger$ where $U' := \det(U)^{1/d} U$ and $\det(U') = 1$. The set of unitary operators with determinant one is known as the *special unitary group* and is denoted by $\mathrm{SU}(d)$.

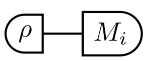
A linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ is invertible if there exists a linear map $\widetilde{C}^{-1} : \mathcal{L}(\mathcal{H}_\mathrm{O}) \to \mathcal{L}(\mathcal{H}_\mathrm{I})$ such that $\widetilde{C}^{-1} \circ \widetilde{C}(\rho) = \rho$ for every $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I})$ and $\widetilde{C} \circ \widetilde{C}^{-1}(\sigma) = \sigma$ for every $\sigma \in \mathcal{L}(\mathcal{H}_\mathrm{O})$. If $\widetilde{C}$ is a quantum channel with inverse map $\widetilde{C}^{-1}$, it follows from Wigner's theorem that $\widetilde{C}^{-1}$ is a quantum channel if and only if $\widetilde{C}$ is unitary [35, 36]. Thus, unitary channels represent reversible quantum operations.

We now introduce the concept of quantum measurements, objects that allow one to extract classical information from quantum states.

Definition 4 (Quantum measurement)**.** *A set of positive linear operators* $\{M_i\}_i$ *is a positive operator-valued measure (POVM) if* $M_i \geq 0$ *and* $\sum_i M_i = \mathbb{1}$.

Quantum measurements are described by POVMs. If we perform a measurement with POVM $\{M_i\}_i$ *on a state* ρ*, the probability of obtaining an outcome* i *is given by*

$$p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i) \tag{2.28}$$

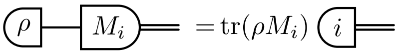
= 

(2.29)

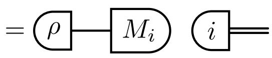
[6]One may wonder why quantum channels are required to be completely positive but not "completely TP". That is, why we do not impose that if $\widetilde{C}$ is a quantum channel, for any linear space $\mathcal{H}_\mathrm{A}$, and any bipartite linear operator $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{A})$, it holds that $\mathrm{tr}\Big(\widetilde{C} \otimes \widetilde{\mathbb{1}}(B)\Big) = \mathrm{tr}(B)$. The reason is straightforward, a linear map is completely TP if and only if it is TP.

[7]In order to ensure that the operator compositions $UU^\dagger$ and $U^\dagger U$ are well defined we implicitly assume that $\mathcal{H}_\mathrm{I}$ and $\mathcal{H}_\mathrm{O}$ are isomorphic.

A quantum measurement $\{M_i\}_i$ can be seen as an operation that transforms a quantum state into a classical state i, which is the label of the POVM. In circuit notation, classical systems are often described by two parallel lines. For instance, the circuit below transforms the quantum state ρ into a classical symbol i with probability $p(i|\rho, \{M_i\}_i) = \mathrm{tr}(\rho M_i)$,



(2.30)



(2.31)

Finally, we described the most general quantum transformation, mathematically formalised by *quantum instruments*. While deterministic quantum operations are described by quantum channels, probabilistic quantum transformations are given by quantum instruments. A quantum instrument may be viewed as an operation that takes as input a quantum state ρ and outputs a classical outcome i and a quantum outcome ρ_i according to some probability distribution.

Definition 5 (Quantum instrument)**.** *A set of linear maps $\left\{\widetilde{C_i}\right\}_i$ is a quantum instrument if $\widetilde{C_i} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ is CP and $\widetilde{C} := \sum_i \widetilde{C_i}$ is CPTP.*

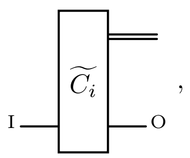
Quantum probabilistic transformations are described by quantum instruments. If we perform a quantum instrument $\left\{\widetilde{C_i}\right\}_i$ on a state ρ, with probability

$$p\left(i\middle|\rho, \left\{\widetilde{C_i}\right\}_i\right) = \mathrm{tr}\left(\widetilde{C_i}(\rho)\right), \tag{2.32}$$

we obtain the outcome i, and the quantum state ρ is transformed into

$$\rho_i = \frac{\widetilde{C_i}(\rho)}{\mathrm{tr}\left(\widetilde{C_i}(\rho)\right)}. \tag{2.33}$$

In circuit notation, an instrument element $\widetilde{C_i} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ can be described by



, (2.34)

where the upper wire indicates the classical outcome.

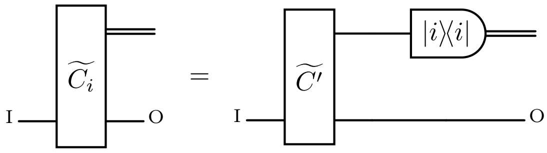
Any quantum instrument can be realised by first applying a quantum channel and then performing a quantum measurement on an auxiliary system. That is, let $\widetilde{C_i} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ be linear CP maps so that $\left\{\widetilde{C_i}\right\}_{i=1}^N$ is a quantum instrument with N outcomes. We can now define the quantum channel $\widetilde{C'} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O} \otimes \mathcal{H}_\mathrm{A})$, where $\mathcal{H}_\mathrm{A} \cong \mathbb{C}^N$ via

$$\widetilde{C'}(\rho) := \sum_{i=1}^N \widetilde{C_i}(\rho) \otimes |i\rangle\langle i|\,. \tag{2.35}$$

It can be verified that for any linear operator $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I})$, we have that

$$\widetilde{C}_i(\rho) = \mathrm{tr}_\mathrm{A}\left[\widetilde{C'}(\rho)\,(\mathbb{1}_\mathrm{O} \otimes |i\rangle\langle i|_\mathrm{A})\right], \tag{2.36}$$

that is, we always have



, (2.37)

Hence, if we apply the channel $\widetilde{C'} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O} \otimes \mathcal{H}_\mathrm{A})$ on an arbitrary state $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I})$, followed by a measurement in the computational basis (i.e., a POVM given by $\{|i\rangle\langle i|\}_{i=1}^{N}$) on the auxiliary system, then, with probability

$$p\left(i\middle|\rho, \left\{\widetilde{C}_i\right\}_i\right) = \mathrm{tr}\left(\widetilde{C}_i(\rho)\right), \tag{2.38}$$

we obtain the outcome i, and the quantum state ρ is transformed into

$$\rho_i = \frac{\widetilde{C}_i(\rho)}{\mathrm{tr}\left(\widetilde{C}_i(\rho)\right)}. \tag{2.39}$$

2.3 Basic concepts for higher-order quantum information

In mathematics and computer science, a higher-order function is a function that takes a function as an input and outputs another function. This concept plays a key role in functional programming, a programming paradigm that inspired higher-order quantum operations. In what follows, we introduce the concept of linear supermaps, mathematical objects that can be interpreted as higher-order functions, as they are linear functions acting on linear maps.

2.3.1 Linear supermaps

Linear supermaps are linear transformations between linear maps. For instance,

$$\widetilde{\widetilde{S}} : \left[\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})\right] \to \left[\mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})\right] \tag{2.40}$$

describes a linear function that transforms an arbitrary linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{I})$ into $\widetilde{\widetilde{S}}\left(\widetilde{C}\right) : \mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})$. Here, the letters P and F in $\mathcal{H}_P$ and $\mathcal{H}_F$ stand for "past" and "future", a convention used in some higher-order quantum operation articles [37]. In this thesis, we will denote supermaps with a double-tilde for clarity.

As we will discuss later, for any supermap $\widetilde{\widetilde{S}} : \left[\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})\right] \to \left[\mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})\right]$, there exists an auxiliary linear space $\mathcal{H}_\mathrm{A}$ and linear maps $\widetilde{E} : \mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{A})$ and $\widetilde{D} : \mathcal{L}(\mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{O}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})$ such that for every linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ it holds that

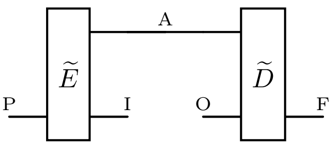
$$\widetilde{\widetilde{S}}(\widetilde{C}) = \widetilde{D} * \widetilde{C} * \widetilde{E} \tag{2.41}$$

$$= \widetilde{D} \circ \left(\widetilde{C} \otimes \widetilde{\mathbb{1}}_A\right) \circ \widetilde{E}. \tag{2.42}$$

For reasons that should become clear soon, the linear maps $\widetilde{E}$ and $\widetilde{D}$ are respectively referred to as encoder and decoder. Note that here, the linear maps $\widetilde{E}$ and $\widetilde{D}$ are not necessarily quantum channels, that is, they are not required to be CP and/or TP. The fact that every linear supermap can be decomposed as a composition of two linear maps allows us to have a convenient description of supermaps in the circuit notation.

In a circuit notation, we can then describe an arbitrary supermap $\widetilde{\widetilde{S}}$ as

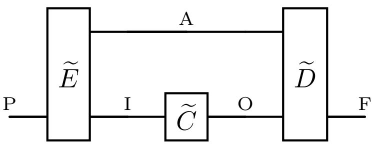
$\widetilde{\widetilde{S}} = \widetilde{E} * \widetilde{D} =$



, (2.43)

and the action of a linear supermap $\widetilde{\widetilde{S}}$ on a linear map $\widetilde{C}$ is described by

$\widetilde{\widetilde{S}}(\widetilde{C}) =$



. (2.44)

Since linear maps have a one-to-one correspondence to bipartite operators via de Choi representation, linear supermaps have a one-to-one correspondence with linear maps. That is, let $\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})] \to [\mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})]$ be a linear supermap and $\widetilde{C} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ an arbitrary linear map. If $C \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O})$, is the Choi operator of $\widetilde{C}$, there exists a linear map $\widetilde{S}$: such that $\widetilde{S}(C)$ is the Choi operator of $\widetilde{\widetilde{S}}(\widetilde{C})$. This isomorphism between linear supermaps and linear operators is described in details in Def. 6 and pictorially illustrated in Eq. (2.49).

Definition 6 (Choi operator of a supermap)**.** *Let* $\mathcal{C}_{\mathrm{I/O}} : [\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})] \to [\mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_\mathrm{O})]$ *be a (invertible) one-to-one function that transforms linear maps* $\widetilde{A} : [\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})]$ *into its Choi operators* $A \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_\mathrm{O})$*, that is* $\mathcal{C}_{\mathrm{I/O}}(\widetilde{A}) = A$ *and* $\mathcal{C}^{-1}_{\mathrm{I/O}}(A) = \widetilde{A}$*. The Choi map* $\widetilde{S} : \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O}) \to \mathcal{L}(\mathcal{H}_\mathrm{P} \otimes \mathcal{H}_\mathrm{F})$ *of a supermap* $\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})] \to [\mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})]$ *is defined as*

$$\widetilde{S} \coloneqq \mathcal{C}_{\mathrm{I/O}} \circ \widetilde{\widetilde{S}} \circ \mathcal{C}^{-1}_{\mathrm{I/O}}. \tag{2.45}$$

The Choi operator of a supermap $\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})] \to [\mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})]$ *is a linear operator* $S \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O} \otimes \mathcal{H}_\mathrm{P} \otimes \mathcal{H}_\mathrm{F})$ *defined as*

$$S \coloneqq \mathcal{C}_{\mathrm{IO/PF}} \circ \mathcal{C}_{\mathrm{I/O}} \circ \widetilde{\widetilde{S}} \circ \mathcal{C}^{-1}_{\mathrm{I/O}}. \tag{2.46}$$

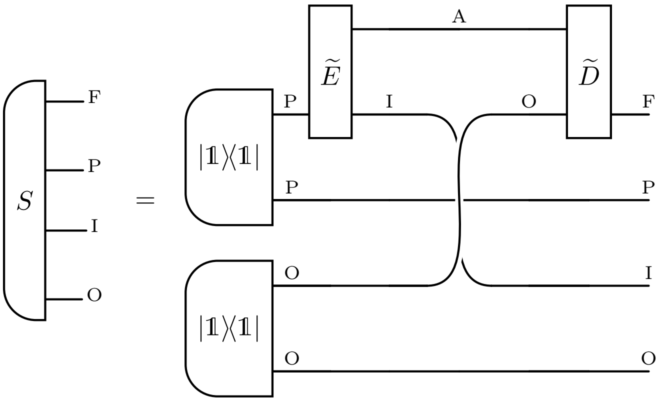
As standard in the field, we often reorder the spaces of S *as* $\mathcal{H}_{\mathrm{P}} \otimes \mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{O}} \otimes \mathcal{H}_{\mathrm{F}}$ *instead of* $H_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{O}} \otimes \mathcal{H}_{\mathrm{P}} \otimes \mathcal{H}_{\mathrm{F}}$ *and write* S_{PIOF}*.*

Using these definitions, the Choi operator of the linear map $\widetilde{\widetilde{S}}(\widetilde{C})$ *is*

$$\widetilde{S}(C) = S_{\mathrm{PIOF}} * A_{\mathrm{IO}} \tag{2.47}$$

$$= \mathrm{tr}_{\mathrm{IO}} \Big(S_{\mathrm{PIOF}} \left[\mathbb{1}_{\mathrm{P}} \otimes C_{\mathrm{IO}}^{T} \otimes \mathbb{1}_{\mathrm{F}} \right] \Big). \tag{2.48}$$

Direct calculation shows that, in circuit notation, the Choi operator S of a supermap $\widetilde{\widetilde{S}}$ reads as



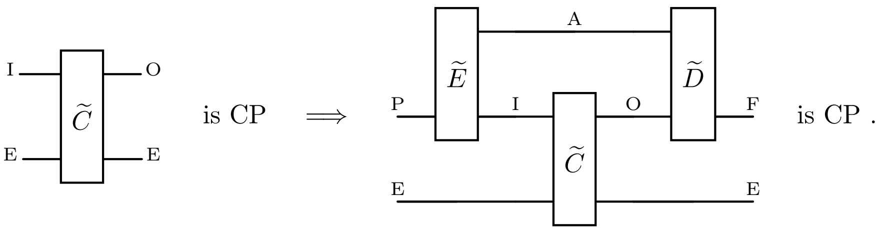
. (2.49)

In the following, we will see that representing linear supermaps by their Choi operators will not only help us to characterise linear supermaps, but it will greatly simplify some calculations.

2.3.2 Quantum superchannels

A linear supermap $\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_{\mathrm{I}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}})] \to [\mathcal{L}(\mathcal{H}_{\mathrm{P}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{F}})]$ is TP preserving if it transforms TP maps into TP maps. That is, if $\widetilde{C} : \mathcal{L}(\mathcal{H}_{\mathrm{I}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{I}})$ is TP, then $\widetilde{\widetilde{S}}\left(\widetilde{C}\right)$ is TP. Similarly, a linear map is CP preserving, if it transforms CP maps into CP maps.

A linear supermap $\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_{\mathrm{I}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}})] \to [\mathcal{L}(\mathcal{H}_{\mathrm{P}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{F}})]$ is completely CP preserving if, for any linear spaces $\mathcal{H}_{\mathrm{E}}$ and any CP linear maps $\widetilde{C} : \mathcal{L}(\mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{E}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}} \otimes \mathcal{H}_{\mathrm{E}})$, the map $\widetilde{\widetilde{S}} \otimes \widetilde{\widetilde{\mathbb{1}}}_{\mathrm{A}}(\widetilde{C})$ is CP, where $\widetilde{\widetilde{\mathbb{1}}}$ is the identity supermap, i.e., $\widetilde{\widetilde{\mathbb{1}}}_{\mathrm{E}}(\widetilde{B}) = \widetilde{B}$ for any $\widetilde{B} : \mathcal{L}(\mathcal{H}_{\mathrm{E}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{E}})$. In circuit notation, $\widetilde{\widetilde{S}} = \widetilde{D} * \widetilde{E}$ is completely CP preserving if for any linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{E}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}} \otimes \mathcal{H}_{\mathrm{E}})$ it holds that



(2.50)

We now define quantum superchannels as operations that transform quantum channels.

Definition 7 (Quantum superchannel)**.** *A linear supermap*

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_\text{I}) \to \mathcal{L}(\mathcal{H}_\text{O})] \to [\mathcal{L}(\mathcal{H}_\text{P}) \to \mathcal{L}(\mathcal{H}_\text{F})] \tag{2.51}$$

is superchannel if $\widetilde{\widetilde{S}}$ *is completely CP preserving and TP preserving.*[8]

We will now present two fundamental result in higher-order quantum operations. These results were published in 2008/2009 in a sequence papers written by Chiribella, D'Ariano, and Perinotti [7, 8, 38], see also [9, 39, 40] for independent related results. Propositions 1 and 2 provide a characterisation of quantum supermaps in terms of positive semidefinite and affine constraints on its Choi operator. Then, Proposition 3 shows that superchannels can always be decomposed as a quantum circuit that may require entanglement with an auxiliary system.

Proposition 1. *A linear supermap*

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_\text{I}) \to \mathcal{L}(\mathcal{H}_\text{O})] \to [\mathcal{L}(\mathcal{H}_\text{P}) \to \mathcal{L}(\mathcal{H}_\text{F})] \tag{2.52}$$

with Choi operator $S \in \mathcal{L}(\mathcal{H}_\text{P} \otimes \mathcal{H}_\text{I} \otimes \mathcal{H}_\text{O} \otimes \mathcal{H}_\text{F})$ *is completely CP preserving if and only if it is positive semidefinite, i.e.,* $S \geq 0$.

Proposition 2. *A linear supermap*

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_\text{I}) \to \mathcal{L}(\mathcal{H}_\text{O})] \to [\mathcal{L}(\mathcal{H}_\text{P}) \to \mathcal{L}(\mathcal{H}_\text{F})] \tag{2.53}$$

with Choi operator $S \in \mathcal{L}(\mathcal{H}_\text{P} \otimes \mathcal{H}_\text{I} \otimes \mathcal{H}_\text{O} \otimes \mathcal{H}_\text{F})$ *is TP preserving if and only if it respects*

$$\text{tr}_\text{F}(S) = \text{tr}_\text{OF}(S) \otimes \frac{\mathbb{1}_\text{O}}{d_\text{O}} \tag{2.54}$$

$$\text{tr}_\text{IOF}(S) = \text{tr}_\text{PIOF}(S) \frac{\mathbb{1}_\text{P}}{d_\text{P}} \tag{2.55}$$

$$\text{tr}(S) = d_\text{P} d_\text{O}. \tag{2.56}$$

where d_i *is the dimension of the linear space* $\mathcal{H}_i$.

Proposition 3. *A linear supermap*

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_\text{I}) \to \mathcal{L}(\mathcal{H}_\text{O})] \to [\mathcal{L}(\mathcal{H}_\text{P}) \to \mathcal{L}(\mathcal{H}_\text{F})] \tag{2.57}$$

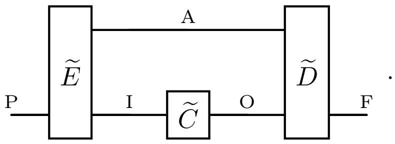
[8] Here, it would make sense to impose that quantum channels are not only TP preserving, but also "completely TP preserving". That is, to impose that for any linear space $\mathcal{H}_\text{A}$, and any linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_\text{I} \otimes \mathcal{H}_\text{A}) \to \mathcal{L}(\mathcal{H}_\text{O} \otimes \mathcal{H}_\text{A})$, the map $\widetilde{\widetilde{S}} \otimes \widetilde{\widetilde{\mathbb{1}}} \left(\widetilde{C}\right)$ is TP. However, it is not hard to show that a linear map is completely TP preserving if and only if it is TP preserving, see e.g., Appendix C (Conditions for validity) of Ref. [37].

is a quantum superchannel if and only if there exists an auxiliary linear space $\mathcal{H}_\mathrm{A}$ *and quantum channels* $\widetilde{E}:\mathcal{L}(\mathcal{H}_\mathrm{P})\to\mathcal{L}(\mathcal{H}_\mathrm{I}\otimes\mathcal{H}_\mathrm{A})$ *and* $\widetilde{D}:\mathcal{L}(\mathcal{H}_\mathrm{A}\otimes\mathcal{H}_\mathrm{O})\to\mathcal{L}(\mathcal{H}_\mathrm{F})$ *such that, for every linear map* $\widetilde{C}:\mathcal{L}(\mathcal{H}_\mathrm{I})\to\mathcal{L}(\mathcal{H}_\mathrm{O})$*, it holds that*

$$\widetilde{\widetilde{S}}(\widetilde{C}) = \widetilde{D} * \widetilde{C} * \widetilde{E} \tag{2.58}$$

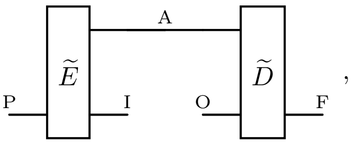
$$= \widetilde{D} \circ \left(\widetilde{C} \otimes \widetilde{\mathbb{1}}_A\right) \circ \widetilde{E} \tag{2.59}$$

$$= \tag{2.60}$$



That is,

$$\widetilde{\widetilde{S}} = \tag{2.61}$$



and the Choi operator of any superchannel $\widetilde{\widetilde{S}}$ *can be decomposed as*[9]

$$S_\mathrm{PIOF} = E_\mathrm{P/IA} * D_\mathrm{AO/F}, \tag{2.62}$$

where E *is the Choi operator of the encoder channel* $\widetilde{E}$ *and* D *is the Choi operator of a decoder channel* $\widetilde{D}$*.*

Also, if $\widetilde{\widetilde{S}}$ *is a superchannel, the Choi operator of* $\widetilde{\widetilde{S}}(\widetilde{C})$ *is given by*

$$S_\mathrm{PIOF} * C_\mathrm{IO} = E_\mathrm{P/IA} * C_\mathrm{I/O} * D_\mathrm{AO/F}. \tag{2.63}$$

The proof of Proposition 3 can be viewed as a generalisation of the Stinespring dilation of quantum channels [8].

2.3.3 Semidefinite programming

The constraints presented in Proposition 1 are all affine or positive semidefinite constraints. This allows us to analyse some optimisation problems involving quantum superchannels as a *semidefinite program* (SDP). An SDP is an optimisation problem where a linear objective function is subjected to positive semidefinite and affine constraints. This class of optimisation problems has been intensively studied in mathematics and computer science, and in addition to various mathematical results, powerful computational methods to solve such problems numerically exist. [41, 42].

[9] To subscript P/IA in $E_\mathrm{P/IA}$ is written in this way to remind us that E is the Choi operator of a linear map transforming operators acting in $\mathcal{H}_\mathrm{P}$ to operators acting in $\mathcal{H}_\mathrm{I}\otimes\mathcal{H}_\mathrm{A}$. In some cases we may use this notation so that we can easily identify the domain and codomain of linear maps described by Choi operators.

Without loss of generality, an SDP can always be written as[10]

$$\text{given:} \quad A \in \mathcal{L}\big(\mathbb{C}^d\big),\ A = A^\dagger \tag{2.64}$$

$$B \in \mathcal{L}\big(\mathbb{C}^{d'}\big),\ B = B^\dagger \tag{2.65}$$

$$\widetilde{\Lambda} : \mathcal{L}\big(\mathbb{C}^d\big) \to \mathcal{L}\big(\mathbb{C}^{d'}\big),\ \widetilde{\Lambda} \text{ is Hermitian preserving} \tag{2.66}$$

$$\max_{\rho} : \quad \mathrm{tr}(A\rho) \tag{2.67}$$

$$\text{subject to:} \quad \Lambda(\rho) = B \tag{2.68}$$

$$\rho \geq 0. \tag{2.69}$$

Linear operators $\rho \in \mathcal{L}(\mathbb{C}^d)$ that respect the constraints in Eq. (2.68) and inequality (2.69) are called feasible points. Any feasible point in a maximisation problem provides a lower bound for the maximum value of the linear objective function.

Associated to any maximisation problem (including optimisation problems that are not SDPs), there exist a minimisation *dual problem*. Using the Lagrange multipliers method, it can be shown that any feasible point for the dual minimisation problem provides an upper bound for the original maximisation problem. Also, if the initial maximisation problem is an SDP (often referred to as the primal problem), the dual problem is also an SDP. Finally, various SDPs appearing in quantum information respect strong duality, which means that the minimum obtained in the dual problem equals the maximum of the primal problem. Later in this thesis we will make use of SDP duality to analyse optimal transformation and discrimination between quantum channels.

2.3.4 Quantum supermeasurements (quantum testers)

We now introduce an abstract framework for performing quantum measurements on quantum channels.

Definition 8 (Quantum supermeasurements (testers))**.** *A set of linear operators* $\{T_i\}_i$, $T_i \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O})$ *is called a tester if*[11] $T_i \geq 0$ *and for any quantum channel* $\widetilde{C} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ *with Choi operator* $C \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_O)$*, the quantity* $\mathrm{tr}(C\,T_i)$ *is a probability distribution on* i*.*

Quantum supermeasurements are described by testers. If we perform a supermeasurement with a tester $\{T_i\}_i$ *on a quantum channel* $\widetilde{C} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$ *with Choi operator* $C \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_O)$*, the probability of obtaining an outcome* i *is given by*

$$p\left(i|\widetilde{C}, \{T_i\}_i\right) = \mathrm{tr}(C\,T_i). \tag{2.70}$$

[10] A linear map is Hermitian preserving if it transforms self-adjoint operators into self-adjoint operators.

[11] Here, we impose T_i to be positive semidefinite because this is equivalent to writing that $\mathrm{tr}(T_i\rho) \geq 0$ for any positive semidefinite matrix ρ. This ensures that when testers are applied to instruments they also lead to non-negative probabilities.

As proven in Refs. [8, 43], a set of linear operators $\{T_i\}_i$, $T_i \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O})$ is tester if and only if

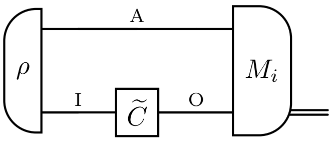
$$T_i \geq 0 \tag{2.71}$$

$$\sum_i T_i = \sigma_\mathrm{I} \otimes \mathbb{1}_\mathrm{O}, \tag{2.72}$$

where $\sigma \in \mathcal{L}(\mathcal{H}_\mathrm{I})$ is an arbitrary quantum state.

Analogous to the realisation theorem for quantum superchannels presented in Proposition 3, Refs. [8, 43] showed that quantum testers also have a realisation in terms of quantum circuits. That is, a set of linear operators $\{T_i\}_i$, $T_i \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O})$ is a tester if and only if there exists a linear space $\mathcal{H}_\mathrm{A}$, a bipartite quantum state $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_A)$, and a POVM $M_i \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_O)$ such that $T_i = \rho_\mathrm{IA} * M_{i,\mathrm{AO}}$, that is

$\mathrm{tr}(CT_i) =$



. (2.73)

We can then see that the act of performing measurements on quantum channels is precisely described by quantum testers.

2.3.5 Quantum superinstruments

Probabilistic transformations between quantum channels are described by quantum superinstruments.

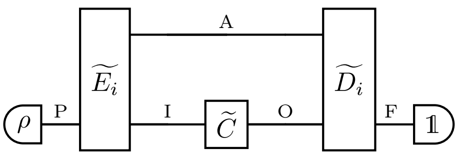
Definition 9 (Quantum superinstrument)**.** *A set of linear supermaps $\{\widetilde{\widetilde{S}}_i\}_i$ is a quantum superinstrument if $\widetilde{\widetilde{S}}_i$ is completely CP preserving and $\widetilde{\widetilde{S}} := \sum_i \widetilde{\widetilde{S}}_i$ is TP preserving.*

If we perform the superinstrument $\{\widetilde{\widetilde{S}}_i\}_i$ on channel $\widetilde{C}$ and state ρ, with probability[12]

$$p(i|\rho, \{\widetilde{\widetilde{S}}_i\}_i) = \mathrm{tr}\left(\left[\widetilde{\widetilde{S}}_i\left(\widetilde{C}\right)\right](\rho)\right) \tag{2.74}$$

$$= \mathrm{tr}(S_i * C * \rho) \tag{2.75}$$

$=$



, (2.76)

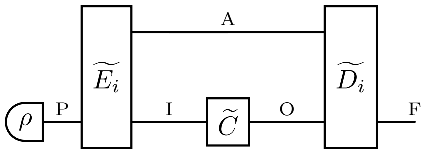
[12]Here we use the notation $\widetilde{\widetilde{S}}_i = \widetilde{E}_i * \widetilde{D}_i$, see Eq. (2.43).

we obtain the outcome i, and the state

$$\rho_i = \frac{\left[\widetilde{\widetilde{S}}_i\left(\widetilde{C}\right)\right](\rho)}{\operatorname{tr}\left(\left[\widetilde{\widetilde{S}}\left(\widetilde{C}\right)\right](\rho)\right)} \tag{2.77}$$

$$= \frac{S_i * C * \rho}{\operatorname{tr}(S_i * C * \rho)} \tag{2.78}$$

$$= \quad . \tag{2.79}$$



Similarly to quantum instruments, any quantum superinstrument can be realised by first applying a quantum superchannel and then performing a quantum measurement on an auxiliary system. That is, let $\widetilde{\widetilde{S}}_i : \Big[\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})\Big] \to \Big[\mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})\Big]$ be linear supermaps so that $\Big\{\widetilde{\widetilde{S}}_i\Big\}_{i=1}^N$ forms a quantum instrument with N outcomes. We can now define the quantum superchannel $\widetilde{\widetilde{S'}} : \Big[\mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})\Big] \to \Big[\mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})\Big] \otimes \mathcal{L}(\mathcal{H}_\mathrm{A})$, with $\mathcal{H}_\mathrm{A} \cong \mathbb{C}^N$, via

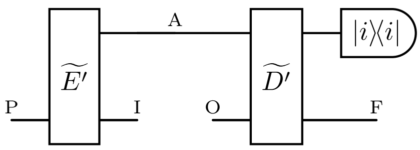
$$\widetilde{\widetilde{S'}}(\widetilde{C}) := \sum_{i=1}^{N} \widetilde{\widetilde{S}}_i(\widetilde{C}) \otimes |i\rangle\!\langle i| \,. \tag{2.80}$$

It can be verified that for any linear map $\widetilde{C} \in \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$, and any operator $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I})$, we have that

$$\left[\widetilde{\widetilde{S}}_i(\widetilde{C})\right](\rho) = \operatorname{tr}_\mathrm{A}\left(\left[\widetilde{\widetilde{S'}}(\widetilde{C})\right](\rho)\ \left[\mathbb{1}_\mathrm{O} \otimes |i\rangle\!\langle i|_\mathrm{A}\right]\right). \tag{2.81}$$

If we decompose the superchannel $\widetilde{\widetilde{S'}}$ as $\widetilde{\widetilde{S'}} = \widetilde{E'} * \widetilde{D'}$, with $\widetilde{E'}$ and $\widetilde{D'}$ being quantum channels, we see that every superinstrument element $\widetilde{\widetilde{S}}_i$ can be decomposed as

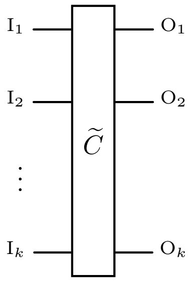
$$\widetilde{\widetilde{S}}_i \quad = \quad . \tag{2.82}$$



Hence, a superinstrument is just a superchannel followed by a quantum measurement on an auxiliary system.

2.3.6 Multipartite channels

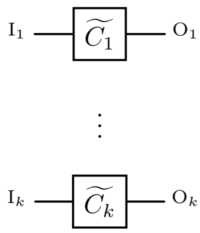
A multipartite quantum channel is a quantum channel whose inputs and outputs have a tensor product structure. That is, it takes k-partite quantum states as inputs, and outputs a k-partite quantum state. In pictorial quantum circuit notation, this is represented by a channel that has k wires as inputs and k wires as outputs. More rigorously, a quantum channel $\widetilde{C} : \mathcal{L}(\mathcal{H}_{\mathrm{I}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}})$ is a k-partite quantum channel if the linear spaces $\mathcal{H}_I$ and $\mathcal{H}_O$ have a multipartite tensor product decomposition given by $\mathcal{H}_I = \bigotimes_{i=1}^{k} \mathcal{H}_{\mathrm{I}_i}$ and $\mathcal{H}_O = \bigotimes_{i=1}^{k} \mathcal{H}_{\mathrm{O}_i}$. In circuit notation we have



$$: \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i}) \tag{2.83}$$

A simple and important class of k-partite channels is when we have k independent channels given by $\widetilde{C_i} : \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})$. Such channels are mathematically described by the tensor product of quantum channels,

$$\bigotimes_{i=1}^{k} \widetilde{C_i} = \tag{2.84}$$

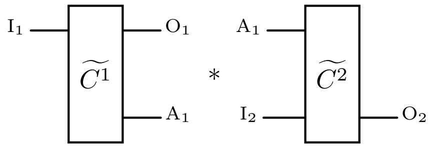


An interesting case considered in this thesis is when all channels $\widetilde{C_i}$ are identical, i.e, $\widetilde{C_i} = \widetilde{C_1}$ for all i. This case admits the interpretation of k calls of the same quantum channel $\widetilde{C_1}$.

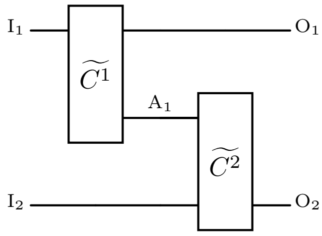
Another very important class of multipartite channels is sequential channels, also referred to as semilocalisable channels [44] (proven to be equivalent to semi-causal channels in Ref. [39]), or channels with memory [10]. Before presenting the general definition, let us start with the bipartite case. A bipartite sequential channel is a quantum channel $\widetilde{C} : \mathcal{L}(\mathcal{H}_{\mathrm{I}_1} \otimes \mathcal{H}_{\mathrm{I}_2}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}_1} \otimes \mathcal{H}_{\mathrm{O}_2})$ that can be decomposed as a sequence of channels $\widetilde{C^1} : \mathcal{L}(\mathcal{H}_{I_1}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}_1} \otimes \mathcal{H}_{\mathrm{A}_1})$ and

$\widetilde{C^2} : \mathcal{L}(\mathcal{H}_{\mathrm{A}_1} \otimes \mathcal{H}_{\mathrm{I}_2}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}_2})$ as

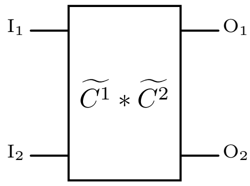
$$\widetilde{C} = \widetilde{C^1} * \widetilde{C^2} \tag{2.85}$$

= 

(2.86)

= 

(2.87)

= 

, (2.88)

with Choi operator given by

$$C = C^1_{\mathrm{I}_1/\mathrm{O}_1\mathrm{A}_1} * C^2_{\mathrm{A}_1\mathrm{I}_2/\mathrm{O}_2}. \tag{2.89}$$

Note that Eq. (2.89) is formally equivalent to Eq. (2.62) from Proposition 3, which characterises a quantum superchannel. Thus, bipartite sequential channels and superchannels may have distinct interpretations, but in practical terms, they are mathematically equivalent.

We now present the general definition of "sequential quantum channels".

Definition 10. *A linear map $\widetilde{C} : \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})$ is a k-partite sequential quantum channel if there exists auxiliary linear spaces $\mathcal{H}_{\mathrm{A}_i}$ and quantum channels $\widetilde{C^1} : \mathcal{L}(\mathcal{H}_{\mathrm{I}_1}) \to \mathcal{L}(\mathcal{H}_{\mathrm{A}_1} \otimes \mathcal{H}_{\mathrm{O}_1})$, $\widetilde{C^k} : \mathcal{L}(\mathcal{H}_{\mathrm{I}_k} \otimes \mathcal{H}_{\mathrm{A}_{k-1}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}_k})$, $\widetilde{C^i} : \mathcal{L}(\mathcal{H}_{\mathrm{I}_i} \otimes \mathcal{H}_{\mathrm{A}_{i-1}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{A}_i} \otimes \mathcal{H}_{\mathrm{O}_i})$ for $i \in \{2, \ldots, k-1\}$, such that their associated Choi operators respect*

$$C = C^1_{\mathrm{I}_1/\mathrm{O}_1\mathrm{A}} * C^2_{\mathrm{A}_1\mathrm{I}_2/\mathrm{O}_2\mathrm{A}_2} * \ldots * C^{k-1}_{\mathrm{A}_{k-2}\mathrm{I}_k/\mathrm{O}_k\mathrm{A}_{k-1}} * C^k_{\mathrm{A}_{k-1}\mathrm{I}_k/\mathrm{O}_k}. \tag{2.90}$$

Sequential channels admit a relatively simple characterisation in terms of their Choi operators. Before presenting this characterisation theorem for sequential k-slot superchannels we introduce the *trace-and-replace* map, originally introduced in Ref. [34], which is useful for studying higher-order operations. The trace-and-replace map is a linear map that partially traces the system in space $\mathcal{H}_i$, and replaces it by a normalised identity $\frac{\mathbb{1}_{d_i}}{d_i}$.

Definition 11 (Trace-and-replace notation)**.** *Let $C \in \bigotimes_{j=1}^{k} \mathcal{L}(\mathcal{H}_j)$ be a k-partite linear operator. The trace-and-replace map ${}_i : \bigotimes_{j=1}^{k} \mathcal{L}(\mathcal{H}_j) \to \bigotimes_{j=1}^{k} \mathcal{L}(\mathcal{H}_j)$ is defined as*

$$ {}_iC \coloneqq \mathrm{tr}_i(C) \otimes \frac{\mathbb{1}_i}{d_i} \tag{2.91} $$

where d_i is the dimension of $\mathcal{H}_i$.

Notice that one should keep track of the ordering of the operators, for instance, if $C \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{O})$, ${}_\mathrm{I}C = \frac{\mathbb{1}_\mathrm{I}}{d_\mathrm{I}} \otimes \mathrm{tr}_\mathrm{I}[C]$ and ${}_\mathrm{O}C = \mathrm{tr}_\mathrm{O}[C] \otimes \frac{\mathbb{1}_\mathrm{O}}{d_\mathrm{O}}$.

We remark that the trace-and-replace map is also equivalent to performing a completely depolarising map on the linear space $\mathcal{H}_i$. That is, let $\widetilde{D} : \mathcal{L}(\mathcal{H}_i) \to \mathcal{L}(\mathcal{H}_i)$ be the completely depolarising map, i.e., $\widetilde{D}(\rho) = \mathrm{tr}(\rho)\frac{\mathbb{1}_i}{d_i}$ for any $\rho \in \mathcal{L}(\mathcal{H}_i)$. It holds that

$$ {}_iC_{12\ldots k} = \left[\widetilde{\mathbb{1}_1} \otimes \ldots \otimes \widetilde{D_i} \otimes \ldots \otimes \widetilde{\mathbb{1}_k}\right](C). \tag{2.92} $$

For instance, if $k = 3$ and $i = 2$, we have that

$$ {}_2C = \left[\widetilde{\mathbb{1}_1} \otimes \widetilde{D_2} \otimes \widetilde{\mathbb{1}_3}\right](C_{123}) = \mathrm{tr}_2(C_{123}) \otimes \frac{\mathbb{1}_2}{d_2}. \tag{2.93} $$

Also, as mentioned before, a linear map $\widetilde{C} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{I})$ is TP if and only if $\mathrm{tr}_\mathrm{O}(C) = \mathbb{1}_\mathrm{I}$. Using the trace-and-replace notation, the TP condition is equivalent to ${}_\mathrm{O}C = {}_\mathrm{IO}C$ and $\mathrm{tr}(C) = d_\mathrm{I}$.

We now present a characterisation of sequential k-partite quantum channels. This result appeared in the literature various times, from different motivations, but essentially the same mathematics behind, e.g., Refs. [7–10, 45]. Moreover, as we will see it in the next subsection, this result has a very close relationship to Proposition 5.

Proposition 4. *A linear map $\widetilde{C} : \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})$ is a k-partite sequential quantum channel if and only if its Choi operator C respects*

$$ C \geq 0 \tag{2.94} $$

$$ {}_{\mathrm{O}_k}C = {}_{\mathrm{I}_k\mathrm{O}_k}C \tag{2.95} $$

$$ {}_{\mathrm{O}_{k-1}\mathrm{I}_k\mathrm{O}_k}C = {}_{\mathrm{I}_{k-1}\mathrm{O}_{k-1}\mathrm{I}_k\mathrm{O}_k}C \tag{2.96} $$

$$ \vdots \tag{2.97} $$

$$ {}_{\mathrm{O}_1\ldots\mathrm{I}_{k-1}\mathrm{O}_{k-1}\mathrm{I}_k\mathrm{O}_k}C = {}_{\mathrm{I}_1\mathrm{O}_1\ldots\mathrm{I}_{k-1}\mathrm{O}_{k-1}\mathrm{I}_k\mathrm{O}_k}C \tag{2.98} $$

$$ \mathrm{tr}(C) = d_{\mathrm{I}_1} d_{\mathrm{I}_2} \ldots d_{\mathrm{I}_k}. \tag{2.99} $$

2.3.7 Multi-slot superchannels

We can now define multi-slot superchannels. We start by defining a parallel k-slot superchannel, which is a superchannel that transforms an arbitrary k-partite quantum channel into a single quantum channel that is essentially equivalent to a (zero-slot) superchannel.

Definition 12 (Parallel k-slot superchannel)**.** *A linear supermap*

$$\widetilde{\widetilde{S}} : \left[\bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})\right] \to \left[\mathcal{L}(\mathcal{H}_{\mathrm{P}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{F}})\right] \tag{2.100}$$

is a k-slot parallel superchannel if $\widetilde{\widetilde{S}}$ is completely CP preserving and for every k-partite quantum channel

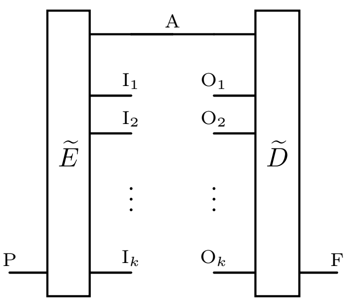
$$\widetilde{C} : \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i}), \tag{2.101}$$

the output linear map $\widetilde{\widetilde{S}}(\widetilde{C})$ is a quantum channel.

Mathematically, a parallel k-slot superchannel is just a standard (1-slot) superchannel with an additional tensor product structure. It follows from Proposition 3 that $\widetilde{\widetilde{S}} : \left[\bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})\right] \to \left[\mathcal{L}(\mathcal{H}_{\mathrm{P}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{F}})\right]$ is a parallel multipartite quantum channel if and only if its Choi operator can be decomposed as

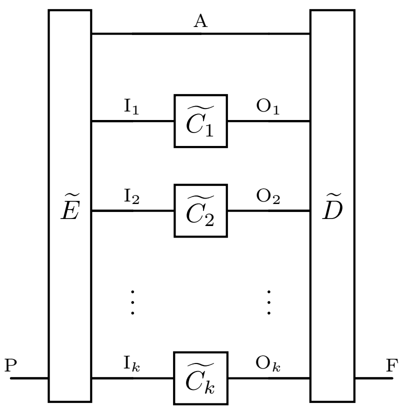
$$S_{\mathrm{PIOF}} = E_{\mathrm{P/IA}} * D_{\mathrm{AO/F}}, \tag{2.102}$$

where $\mathcal{H}_I \coloneqq \bigotimes_{i=1}^{k} \mathcal{H}_{\mathrm{I}_i}$ and $\mathcal{H}_O \coloneqq \bigotimes_{i=1}^{k} \mathcal{H}_{\mathrm{O}_i}$ and E and D are Choi operators of quantum channels. Hence, a parallel superchannel can always be written as

$\widetilde{\widetilde{S}}^{\mathrm{PAR}} =$  , (2.103)

where $\widetilde{E}$ and $\widetilde{D}$ are quantum channels.

Applying a parallel superchannel to k independent quantum channels results in

$\widetilde{\widetilde{S}}^{\mathrm{PAR}}\left(\bigotimes_{i=1}^{k} \widetilde{C}_i\right) =$  . (2.104)

Additionally, if all channels $\widetilde{C_i}$ are the same, e.g., $\widetilde{C_i} = \widetilde{C_1}$ for every i, a k-slot parallel superchannel can be viewed as an operation that makes k parallel calls to the same channel $\widetilde{C_1}$.

We now present the definition of sequential k-slot superchannels, objects that can transform sequential k-partite channels into channels. While any k-partite quantum channel can be "plugged" into a k-slot parallel superchannel, not all k-partite quantum channels can be plugged into sequential k-slot superchannel. In this sequential case, we only impose that sequential k-partite quantum channels are transformed into channels. Since we relax the requirements, the set of sequential superchannels is strictly larger than the set of parallel superchannels.

Sequential superchannels are also referred to in the literature as quantum combs [7, 8], and quantum strategies [9], causally ordered process matrices [28, 34], multi-time quantum processes [45], and quantum circuits with fixed causal order [46].

Definition 13 (Sequential k-slot superchannel)**.** *A linear supermap*

$$\widetilde{\widetilde{S}} : \left[\bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})\right] \to \left[\mathcal{L}(\mathcal{H}_{\mathrm{P}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{F}})\right] \tag{2.105}$$

is a k-slot sequential superchannel if it is completely CP preserving and for every k-partite sequential quantum channel

$$\widetilde{C} : \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i}), \tag{2.106}$$

the output linear map $\widetilde{\widetilde{S}}(\widetilde{C})$ is a quantum channel.

Similarly to the single-slot case, sequential k-slot superchannels can always be decomposed as a causally ordered circuit [7–9]. Also, a sequential k-slot superchannel is formally equivalent to a $(k+1)$-partite sequential channel [7–10]. These results can be summarised in the proposition below.

Proposition 5. *Let*

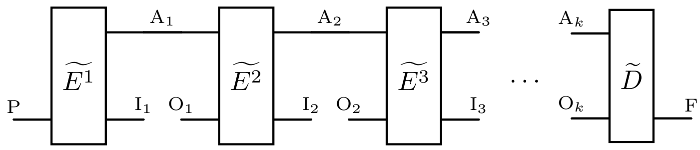
$$\widetilde{\widetilde{S}} : \left[\bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})\right] \to \left[\mathcal{L}(\mathcal{H}_{\mathrm{P}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{F}})\right] \tag{2.107}$$

be a linear supermap with Choi operator S. The following three statements are equivalent.

- *$\widetilde{\widetilde{S}}$ is a k-slot superchannel.*
- *There exist quantum channels $\widetilde{E^1} : \mathcal{L}(\mathcal{H}_{\mathrm{P}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{I}_1} \otimes \mathcal{H}_{\mathrm{A}_1})$, $\widetilde{D} : \mathcal{L}(\mathcal{H}_{\mathrm{O}_k} \otimes \mathcal{H}_{\mathrm{A}_k}) \to \mathcal{L}(\mathcal{H}_{\mathrm{F}})$, and, for $i \in \{2, \dots, k-1\}$, $\widetilde{E^i} : \mathcal{L}(\mathcal{H}_{\mathrm{O}_{i-1}} \otimes \mathcal{H}_{\mathrm{A}_{i-1}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{I}_i} \otimes \mathcal{H}_{\mathrm{A}_i})$,*

such that

$$\tilde{\tilde{S}} = \widetilde{E^1} * \widetilde{E^2} * \ldots \widetilde{D} \tag{2.108}$$

$=$ 

(2.109)

- *The Choi operator S respects*

$$S \geq 0 \tag{2.110}$$
$${}_{\mathrm{F}}S = {}_{\mathrm{O}_k\mathrm{F}}S \tag{2.111}$$
$${}_{\mathrm{I}_k\mathrm{O}_k\mathrm{F}}S = {}_{\mathrm{O}_{k-1}\mathrm{I}_k\mathrm{O}_k\mathrm{F}}S \tag{2.112}$$
$${}_{\mathrm{I}_{k-1}\mathrm{O}_{k-1}\mathrm{I}_k\mathrm{O}_k\mathrm{F}}S = {}_{\mathrm{O}_{k-2}\mathrm{I}_{k-1}\mathrm{O}_{k-1}\mathrm{I}_k\mathrm{O}_k\mathrm{F}}S \tag{2.113}$$
$$\vdots$$
$${}_{\mathrm{I}_1\mathrm{O}_1\ldots\mathrm{I}_k\mathrm{O}_k\mathrm{F}}S = {}_{\mathrm{P}\mathrm{I}_1\mathrm{O}_1\ldots\mathrm{I}_k\mathrm{O}_k\mathrm{F}}S \tag{2.114}$$
$$\mathrm{tr}(S) = d_{\mathrm{P}} d_{\mathrm{O}_1} \ldots d_{\mathrm{O}_k}\ . \tag{2.115}$$

We now present the definition of general k-slot superchannels, objects that can transform k independent channels into a channel. The set of channels "accepted" by general superchannels is very restricted, it only accepts independent channels (that compose under a tensor product)[13].

General superchannels were first introduced in Ref. [27] under the name of deterministic supermaps on no-signalling channels and deterministic supermaps on nonsignalling channels. Motivated by analysing correlations that cannot be explained by a definite causal order, Ref. [28] introduced a bipartite version of general superchannels as process matrices, a concept that was extended to more parties in Ref. [34], in the latter case being equivalent to arbitrary general superchannels. Later, in Ref. [37], general superchannels appeared under the name of process matrices with an open past and open future, using a notation that is closer to the one used here.

Definition 14 (General k-slot superchannel)**.** *A linear supermap*

$$\tilde{\tilde{S}} : \left[\bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \bigotimes_{i=1}^{k} \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})\right] \to \Big[\mathcal{L}(\mathcal{H}_{\mathrm{P}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{F}})\Big] \tag{2.116}$$

is a k-slot general superchannel if it is completely CP preserving for every quantum channels $\widetilde{C_i} : \mathcal{L}(\mathcal{H}_{\mathrm{I}_i}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}_i})$, if

$$\widetilde{C} = \bigotimes_{i=1}^{k} \widetilde{C_i}, \tag{2.117}$$

the output linear map $\tilde{\tilde{S}}(\widetilde{C})$ is a quantum channel.

[13] Due to linearity, any affine combination of independent channels would then be "accepted" by general superchannels. Affine combinations of independent channels form what is called *non-signalling channels* [27]. An alternative equivalent definition of general superchannels is to state that these are linear supermaps transforming nonsignalling channels into channels [27].

Similarly to quantum parallel and sequential superchannels, general superchannels can also be characterised in terms of their Choi operators. The characterisation for arbitrary k can be found in Refs. [34, 47]. When $k = 2$, it can be shown that a supermap $\widetilde{S}$ is a general 2-slot superchannel if and only if its Choi operator S respects

$$S \geq 0 \tag{2.118}$$
$$_{\mathrm{I_2O_2F}}S = {}_{\mathrm{O_1I_2O_2F}}S \tag{2.119}$$
$$_{\mathrm{I_1O_1F}}S = {}_{\mathrm{I_1O_1O_2F}}S \tag{2.120}$$
$$_{\mathrm{PI_1O_1I_2O_2F}}S = {}_{\mathrm{PI_1O_1I_2O_2F}}S \tag{2.121}$$
$$_{\mathrm{F}}S + {}_{\mathrm{O_1O_2F}}S = {}_{\mathrm{O_1F}}S + {}_{\mathrm{O_2F}}S \tag{2.122}$$
$$\mathrm{tr}(S) = d_{\mathrm{F}} d_{\mathrm{O_1}} d_{\mathrm{O_2}}. \tag{2.123}$$

Unlike parallel and sequential superchannels, general k-slot superchannels are not guaranteed to have a realisation in terms of quantum circuits. In various cases, such general superchannels represent processes that do not respect any definite causal order [27, 28]. Some general superchannels such as the quantum switch (see Sec. 4.1.5) are examples of processes without a definite causal order that admit an interpretation as a quantum control of sequential superchannels. However, there exist general superchannels, such as the W_{OCB} from Ref. [28] that cannot be viewed as a quantum control of causal orders [46].

The interpretation of general superchannels remains an area of active debate, as the physics underlying these mathematical constructs is still under development. Although we may not yet have a "fair and concrete physical realisation" for arbitrary general superchannels, we argue that this class of superchannels is worthy of study. First, exploring the power and limitations of general superchannels may lead to valuable insights into the ongoing debate about whether general superchannels, or specific subsets thereof, could have a "fair" physical implementation. Second, since the set of general superchannels includes sequential superchannels, any identified limitations for general superchannels (e.g., an upper bound on performing a specific task) also apply to sequential superchannels. Moreover, the set of general superchannels is permutation invariant over their slots, just as in the parallel case. That is, if $\widetilde{\widetilde{S}}$ is a valid general or parallel k slot superchannel, we can define a new valid general or parallel superchannel $\widetilde{\widetilde{\widetilde{S'}}}$ by permuting slots, that is, permuting spaces $\mathcal{H}_{\mathrm{I}_i} \otimes \mathcal{H}_{\mathrm{O}_i}$. Notice that property not satisfied by sequential case, since permuting the slots may result into a non-valid sequential superchannel. This invariance under slot permutations introduces symmetries that can significantly simplify the solution of certain problems. Thus, even if general superchannels are considered a mathematical abstraction, they remain a powerful tool for identifying limitations and deriving upper bounds in tasks involving sequential superchannels, which are within the standard quantum circuit formalism.

Chapter 3

Transforming unitary channels

This chapter contains original results extracted from the following publications:

- [21] Reversing Unknown Quantum Transformations: Universal Quantum Circuit for Inverting General Unitary Operations
  M.T. Quintino, Q. Dong, A. Shimbo, A. Soeda, M. Murao
  Phys. Rev. Lett. 123, 210502 (2019)
  arXiv:1810.06944 [quant-ph]

- [13] Probabilistic exact universal quantum circuits for transforming unitary operations
  M.T. Quintino, Q. Dong, A. Shimbo, A. Soeda, M. Murao
  Phys. Rev. A 100, 062339 (2019)
  arXiv:1909.01366 [quant-ph]

- [48] Success-or-Draw: A Strategy Allowing Repeat-Until-Success in Quantum Computation
  Q. Dong, **M.T. Quintino**, A. Soeda, M. Murao
  Phys. Rev. Lett. 126, 150504 (2021)
  arXiv:2011.01055 [quant-ph]

- [49] Deterministic transformations between unitary operations: Exponential advantage with adaptive quantum circuits and the power of indefinite causality
  M.T. Quintino, D. Ebler
  Quantum 6, 679 (2022)
  arXiv:2109.08202 [quant-ph]

- [50] Optimal Universal Quantum Circuits for Unitary Complex Conjugation
  D. Ebler, M. Horodecki, M. Marciniak, T. Młynik, **M.T. Quintino**, M. Studziński
  IEEE Trans. Inf. Theory 69, 5069-5082 (2023)
  arXiv:2206.00107 [quant-ph]

- [51] One-to-one Correspondence between Deterministic Port-Based Teleportation and Unitary Estimation
  S. Yoshida, Y. Koizumi, M. Studziński, **M.T. Quintino**, M. Murao
  arXiv:2408.11902 [quant-ph] (2024)

3.1 The problem of transforming unitary operations

In quantum mechanics, deterministic transformations between states are represented by quantum channels and probabilistic transformations by quantum instruments, which consist of quantum channels followed by a quantum measurement. Understanding the properties of quantum channels and quantum instruments is a standard and well established field of research with direct impact for theoretical and applied quantum physics [52, 53]. Similarly to states, quantum channels may also be subjected to universal transformation in a paradigm usually referred as *higher order* transformations. Higher order transformations can be formalised by supermaps [8, 38] and physically implemented by means of quantum circuits. Despite its fundamental value and potential for applications (*e.g.*, quantum circuit designing [7], quantum process tomography [17], testing causal hypothesis [22], channel discrimination [11], aligning reference frames [24, 54], analysing the role of causal order [27, 28, 55]), higher order transformations are still not well understood when compared to quantum channels and quantum instruments.

Reversible operations play an important role in mathematics and in various physical theories such as quantum mechanics and thermodynamics. In quantum mechanics, reversible operations are represented by unitary operators [35, 36]. We focus on *universal* transformations between reversible quantum transformations, that is, we seek for quantum circuits which implement the desired transformation for any unitary operation of some fixed dimension without any further specific details of the input unitary operation. From a practical perspective, this universal requirement ensures that the circuit does not require any readjustments or modification when different inputs are considered and the circuit implements the desired transformation even when the description of the d-dimensional reversible operation is unknown. Note that the universal requirement also imposes strong constraints on transformations which can be physically realised. A well-known example which pinpoints these constraints when considering quantum states is quantum cloning, although it is simple to construct a quantum device that clones qubits which are promised to be in the state $|0\rangle$ or $|1\rangle$, it is not possible to design a universal quantum transformation that clones all qubit states [1]. Another interesting example can be found in Ref. [56] where the authors consider universal not gates for qubits.

In this chapter we consider the task of transforming $k \in \mathbb{N}$ calls of an arbitrary d-dimensional unitary channel $\widetilde{U_d}$ into $\widetilde{f(U_d)}$, where $f : \mathrm{SU}(d) \to \mathrm{SU}(d')$ is a function that transforms d-dimensional operators into d'-dimensional operators. More precisely, we seek a parallel/sequential/general k-slot quantum superchannel

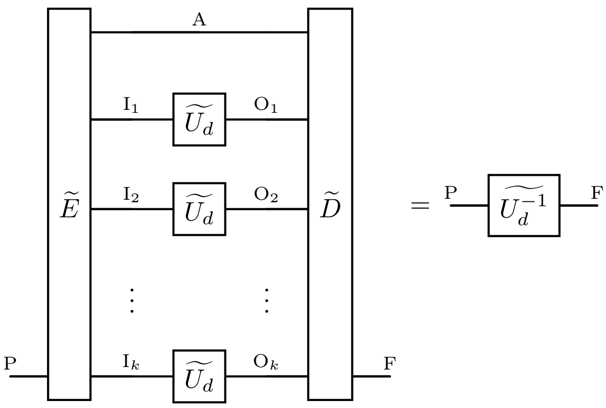
$$\widetilde{\widetilde{S}} : \left[\mathcal{L}(\mathbb{C}^d)^{\otimes k} \to \mathcal{L}(\mathbb{C}^d)^{\otimes k}\right] \to \left[\mathcal{L}(\mathbb{C}^{d'}) \to \mathcal{L}(\mathbb{C}^{d'})\right] \tag{3.1}$$

such that

$$\widetilde{\widetilde{S}}\left(\widetilde{U_d}^{\otimes k}\right) = \widetilde{f(U_d)} \tag{3.2}$$

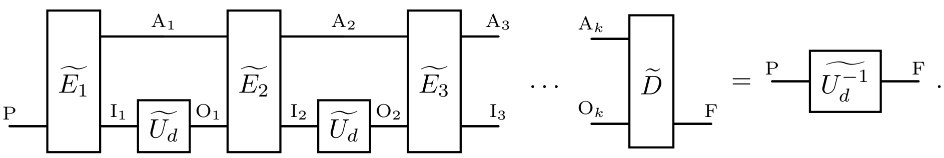
holds for every unitary operator $U_d \in \mathrm{SU}(d)$. For instance, a relevant example that we analyse in this chapter is the case $f(U_d) = U_d^{-1}$. In this case, we seek a superchannel that can reverse the action of a unitary channel.

In the parallel case, we seek channels $\widetilde{E}$ and $\widetilde{D}$ that satisfy



(3.3)

In the sequential case, we seek channels $\widetilde{E_i}$ and $\widetilde{D}$ that satisfy



(3.4)

Unlike the parallel and sequential cases, general superchannels may lack a concrete quantum circuit realisation.

3.1.1 The probabilistic exact approach

For some particular functions $f : \mathrm{SU}(d) \to \mathrm{SU}(d)$ and some fixed number of calls k, there is no k-slot superchannel $\widetilde{\widetilde{S}}$ such that $\widetilde{\widetilde{S}}\left(\widetilde{U_d}^{\otimes k}\right) = \widetilde{f(U_d)}$ in a deterministic way. In such cases, it is natural to consider probabilistic transformations, and to seek superinstruments leading to the highest success probability. More precisely, we seek a parallel/sequential/general k-slot superinstrument $\{\widetilde{\widetilde{S}}, \widetilde{\widetilde{F}}\}$ such that

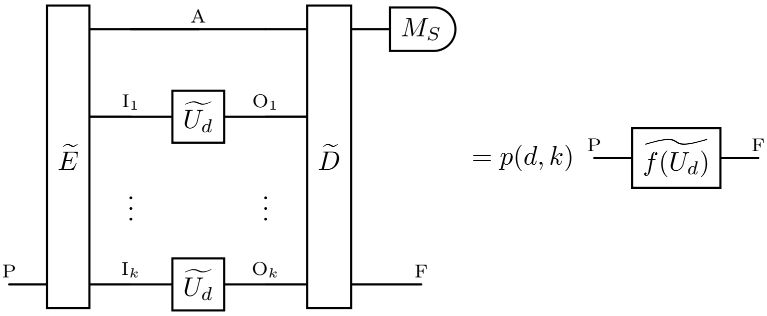
$$\widetilde{\widetilde{S}}\left(\widetilde{U_d}^{\otimes k}\right) = p(d,k)\,\widetilde{f(U_d)}, \tag{3.5}$$

where $p(d,k)$ is the probability of success of implementing the function f using k calls of a unitary channel $\widetilde{U_d}$. Here $\widetilde{\widetilde{S}}$ is a superinstrument element associated to a successful implementation of the transformation, and $\widetilde{\widetilde{F}}$ is an instrument element associated to failure of the desired transformation.[1] In this approach, our goal is then to maximise the success probability $p(d,k)$ under the constraint that $\{\widetilde{\widetilde{S}}, \widetilde{\widetilde{F}}\}$ is a parallel/sequential/general superchannel.

As discussed after the definition of superinstruments (Def. 2.3.5), such probabilistic transformations can always be realised by a superchannel followed

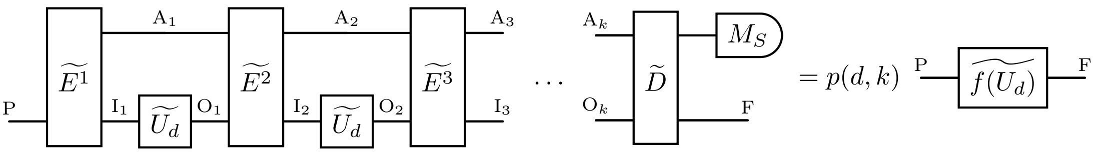
[1] On this approach, the success or failure of the transformation is indicated by a classical outcome provided by the superinstrument. In a quatum physics context, such scenarios are sometimes referred to as probabilstic heralded.

by a quantum measurement on some auxiliary system. For instance, when analysing the parallel probabilistic exact case, we seek quantum channels $\widetilde{E}$ and $\widetilde{D}$ and a POVM $\{M_S, M_F\}$ such that



(3.6)

When analysing the sequential probabilistic exact case, we seek quantum channels $\widetilde{E_i}$ and $\widetilde{D}$ and a POVM $\{M_S, M_F\}$ such that



(3.7)

3.1.2 The deterministic approximate approach

Another approach is to consider deterministic but nonexact transformations. A standard way to quantify the performance in such tasks is via the fidelity between the desired channel and the output channel. The fidelity between a quantum channel $\widetilde{C} : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ and a unitary channel $\widetilde{U_d} : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ is given by

$$F\left(\widetilde{C}, \widetilde{U_d}\right) := \frac{1}{d^2} \operatorname{tr}(C \, |U_d\rangle\!\rangle\langle\!\langle U_d|), \tag{3.8}$$

where C and $|U_d\rangle\!\rangle\langle\!\langle U_d|$ are the respective Choi operators of $\widetilde{C}$ and $\widetilde{U_d}$. The channel fidelity is known to respect various properties [57], and it is connected to quantum state fidelity, $F_\rho(\rho, |\psi\rangle\langle\psi|) := \operatorname{tr}(\rho\, |\psi\rangle\langle\psi|)$, via the mathematical identity [58]

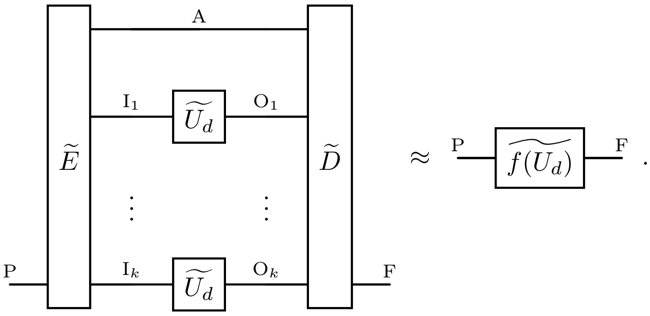
$$F\left(\widetilde{C}, \widetilde{U_d}\right) = \left((d+1)\int_{\text{Haar}} F_\rho\left(\widetilde{C}\left(V\,|\psi\rangle\langle\psi|\,V^\dagger\right), \widetilde{U_d}\left(V\,|\psi\rangle\langle\psi|\,V^\dagger\right)\right) \mathrm{d}V \; - 1\right)\frac{1}{d}, \tag{3.9}$$

where $|\psi\rangle\langle\psi| \in \mathcal{L}(\mathbb{C}^d)$ is an arbitrary rank-1 quantum state, e.g, we can simply take $|\psi\rangle\langle\psi| = |0\rangle\langle 0|$.

We now define the performance $F(d,k)$ of some parallel/sequential/general k-slot superchannel $\widetilde{\widetilde{S}}$ used to implement a function f as the worst-case fidelity, that is,

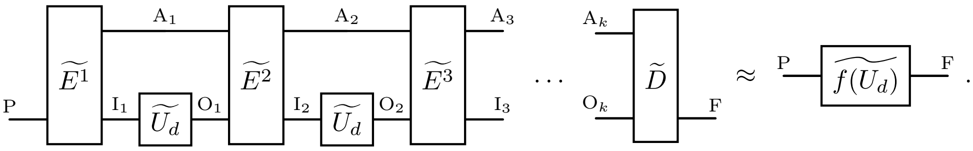
$$F(d,k) := \min_{U_d} F\left(\widetilde{\widetilde{S}}(\widetilde{U_d}^{\otimes k}), \widetilde{f(U_d)}\right). \tag{3.10}$$

When analysing the parallel deterministic case, we seek quantum channels $\widetilde{E}$ and $\widetilde{D}$ such that



(3.11)

When analysing the sequential deterministic case, we seek quantum channels $\widetilde{E}_i$ and $\widetilde{D}$ such that



(3.12)

Another natural way to quantify the performance for some k-slot superchannel to transform U_d into $f(U_d)$ would be to consider the average fidelity, or some robustness based quantifier. In the next section we will see that for the functions considered in this work, all these quantifiers are equivalent, i,e, optimisation of all of these quantifiers leads to the same results.

3.2 Homomorphisms and antihomomorphisms

In this thesis, we reserve the theorem environment exclusively for results originally presented in a publication of the author. This can be contrasted to the proposition environment, which presents results proven in articles not co-written by the author. Each theorem is presented with the original reference as its name (e.g., Theorem 1 (Ref. [49])).

A function $f : \mathrm{SU}(d) \to \mathrm{SU}(d')$ is homomorphic if $f(UV) = f(U)f(V)$ holds for every $U, V \in \mathrm{SU}(d)$, that is, it preserves the group structure. Also, $f : \mathrm{SU}(d) \to \mathrm{SU}(d')$ is antihomomorphic if $f(UV) = f(V)f(U)$ holds for every $U, V \in \mathrm{SU}(d)$. In this chapter we will restrict ourselves only to homomorphic and antihomomorphic functions, since they greatly simplify the analysis.

Theorem 1 (Ref. [49])**.** *Let $f : \mathrm{SU}(d) \to \mathrm{SU}(d')$ be a homomorphic or antihomomorphic function and $\widetilde{\widetilde{S}} : \left[\mathcal{L}(\mathbb{C}^d)^{\otimes k} \to \mathcal{L}(\mathbb{C}^d)^{\otimes k}\right] \to \left[\mathcal{L}(\mathbb{C}^{d'}) \to \mathcal{L}(\mathbb{C}^{d'})\right]$ be a parallel/sequential/general k-slot supermap that transforms $\widetilde{U_d}^{\otimes k}$ to $\widetilde{f(U_d)}$ with an average fidelity given by*

$$F(d,k) := \int_{\mathrm{Haar}} F\Big(\widetilde{\widetilde{S}}(\widetilde{U_d}^{\otimes k}), \widetilde{f(U_d)}\Big)\, dU_d. \tag{3.13}$$

There exists a parallel/sequential/general k-slot superchannel $\widetilde{\widetilde{S'}}$ *such that*

$$F\left(\widetilde{\widetilde{S'}}(\widetilde{U_d}^{\otimes k}), \widetilde{f(U_d)}\right) = F(d,k), \quad \forall U_d \in \mathrm{SU}(d), \tag{3.14}$$

that is, the fidelities for all unitary channels $\widetilde{U_d}$ *are the same, hence the worst-case fidelity is* $\min_{U_d} F\left(\widetilde{\widetilde{S}}(\widetilde{U_d}^{\otimes k}), \widetilde{f(U_d)}\right) = F(d,k)$.

Additionally, if $f(U_d^*) = f(U_d)^*$ *and* $\mathrm{d}U_d = \mathrm{d}f(U_d)$, *it holds that*

$$\widetilde{\widetilde{S'}}(\widetilde{U_d}) = \eta\, \widetilde{f(U_d)} + (1-\eta)\widetilde{D}, \tag{3.15}$$

where $\eta = \frac{F(d,k)d^2-1}{d^2-1}$ *and* $\widetilde{D}(\rho) := \mathrm{tr}(\rho)\frac{\mathbb{1}}{d'}$ *is the complete depolarising channel.*

The proof of this theorem is presented in Ref. [49], and it is based on Holevo covariance arguments [59], a method that has also appeared in the context of higher-order quantum operations (see, e.g., Refs. [24, 55, 60, 61]).

We now state a powerful result first presented in Ref. [61], which proves[2] that when dealing with homomorphic functions, quantum superchannels can be "parallelised" without loss in performance. In particular, if f is a homomorphism, the optimal performance of transforming k calls of $\widetilde{U_d}$ into $\widetilde{f(U_d)}$ is attained by a parallel superchannel.

Proposition 6. *Let* $f : \mathrm{SU}(d) \to \mathrm{SU}(d')$ *be a homomorphic function and* $\widetilde{\widetilde{S}} : \left[\mathcal{L}(\mathbb{C}^d)^{\otimes k} \to \mathcal{L}(\mathbb{C}^d)^{\otimes k}\right] \to \left[\mathcal{L}(\mathbb{C}^{d'}) \to \mathcal{L}(\mathbb{C}^{d'})\right]$ *be a general k-slot supermap superchannel that approximately transforms* $\widetilde{U_d}^{\otimes k}$ *into* $\widetilde{f(U_d)}$ *with worst-case fidelity given by* $F^{\mathrm{GEN}}(d,k)$.

Then, there exists a parallel k-slot superchannel $\widetilde{\widetilde{S^{\mathrm{PAR}}}}$ *such that*

$$F\left(\widetilde{\widetilde{S^{\mathrm{PAR}}}}(\widetilde{U_d}^{\otimes k}), \widetilde{f(U_d)}\right) = F^{\mathrm{GEN}}(d,k), \quad \forall U_d \in \mathrm{SU}(d). \tag{3.16}$$

In the next section of this chapter, we restrict our attention to the case where $d = d'$, that is, f is a function that maps d-dimensional unitaries to d-dimensional unitaries. It is known that [62, 63], up to unitary equivalences, there exist only three homomorphic functions $f : \mathrm{SU}(d) \to \mathrm{SU}(d)$. The trivial function $f(U_d) = \mathbb{1}_d$, the identity function $f(U_d) = U_d$, and the complex conjugation function $f(U_d) = U_d^*$. Additionally, there exist only three antihomomorphic functions $f : \mathrm{SU}(d) \to \mathrm{SU}(d)$, the trivial function $f(U_d) = \mathbb{1}_d$, the transposition function $f(U_d) = U_d^T$, and the inverse function $f(U_d) = U_d^{-1}$. We then restrict our analysis to three non-trivial different functions, unitary complex conjugation, unitary transposition, and unitary inversion.

3.3 Unitary complex conjugation

The possibility of transforming a single call of an arbitrary d-dimensional unitary channel into its complex conjugate was first considered in Ref. [64], where the

[2] In Appendix A of Ref. [49], we revisit this result of Ref. [61] and present an alternative proof using a notation that is closer to the one used in this thesis.

authors showed that the optimal average fidelity is $F_{\mathrm{conj}}(d, k=1) = \frac{2}{d(d-1)}$. Then, Ref. [65] presented a parallel superchannel such that $\widetilde{\widetilde{S}}\left(\widetilde{U_d}^{\otimes(d-1)}\right) = \widetilde{U_d^*}$ for all d-dimensional unitary channels. That is, if $k = d-1$, unitary complex conjugation can be done in a deterministic and exact way.

In Ref. [13] we proved that when $k < d-1$, any probabilistic exact superchannel will necessarily have a zero probability of success.

Theorem 2 (Ref. [13])**.** *Let $\widetilde{\widetilde{S}}$ be a probabilistic k-slot general superchannel (i.e., $\widetilde{\widetilde{S}}$ is an element of a general superinstrument) that transforms k calls of an arbitrary unitary channel $\widetilde{U_d}$ into its complex conjugate, that is,*

$$\widetilde{\widetilde{S}}\left(\widetilde{U_d}^{\otimes k}\right) = p_{\mathrm{conj}}(d,k)\,\widetilde{U_d^*}, \quad \forall U_d \in \mathrm{SU}(d). \tag{3.17}$$

If $k < d-1$, we necessarily have $p_{\mathrm{conj}}(d,k) = 0$.

In Ref. [13] we actually prove a stronger version of Thm. 2, since there we show that even if we restricted ourselves to unitary operators U_d that are diagonal in the computational basis, when $k < d-1$, we necessarily have $p_{\mathrm{conj}}(d,k) = 0$.

In a later work (Ref. [50]), we considered the problem of deterministic unitary complex conjugation and obtained an optimal deterministic, nonexact protocol for transforming k calls of an arbitrary unitary channel $\widetilde{U_d}$ into its complex conjugate $\widetilde{U_d^*}$.

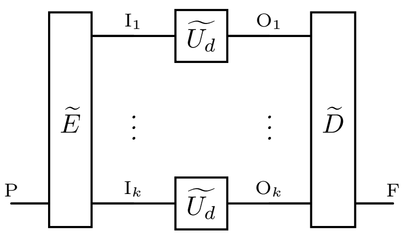
Theorem 3 (Ref. [50])**.** *Let $\widetilde{\widetilde{S}}$ be a deterministic k-slot general superchannel that transforms k calls of an arbitrary unitary channel $\widetilde{U_d}$ into its complex conjugate. The maximum worst-case fidelity is*

$$F_{\mathrm{conj}}(d,k) = \frac{k+1}{d(d-k)}. \tag{3.18}$$

This maximum value is attained by the parallel superchannel $\widetilde{\widetilde{S}}$ that does not make use of an auxiliary system. More precisely, the optimal k-slot superchannel satisfies

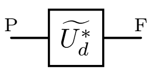
$$\widetilde{\widetilde{S}}(\widetilde{U_d}) = \widetilde{D} \circ \widetilde{U_d}^{\otimes k} \circ \widetilde{E} \tag{3.19}$$

$=$



(3.20)

$\approx$



, (3.21)

where $\widetilde{E} : \mathcal{L}(\mathcal{H}_\mathrm{P}) \to \mathcal{L}(\mathcal{H}_\mathrm{I})$ *and* $\widetilde{D} : \mathcal{L}(\mathcal{H}_\mathrm{O}) \to \mathcal{L}(\mathcal{H}_\mathrm{F})$ *are quantum channels with Choi operators*

$$E := \frac{d}{\binom{d}{k+1}} A_{PI}(d, k+1) \tag{3.22}$$

$$D := \frac{\binom{d}{k}}{\binom{d}{k+1}} A_{OF}(d, k+1) + \Big[\mathbb{1}_\mathrm{O} - A_O(d,k)\Big] \otimes \sigma_\mathrm{F} \tag{3.23}$$

with $A_i(d,k)$ *being the projector onto the antisymmetric subspace*[3] *of* $\mathcal{H}_i$*, where* $\mathcal{H}_i \cong \mathbb{C}^{d^{\otimes k}}$ *and* $\sigma \in \mathcal{L}(\mathcal{H}_\mathrm{F})$ *is an arbitrary quantum state.*

The proof of this theorem is presented in Ref. [50] and can be divided in two parts. First, we verify that the encoder and decoder based on the antisymmetric subspace presented in the statement of Thm. 3 attains the fidelity $F_\mathrm{conj}(d,k) = \frac{k+1}{d(d-k)}$. Then, we make use of the performance operator approach [64] to write the optimisation problem of maximising the worst-case fidelity as an SDP. Then, we analyse the symmetries of the problem. In particular, without loss of generality we can assume the superchannel to be invariant under permutation of the slots, and that the Choi operator of the superchannel respects the commutation relation

$$[S_\mathrm{PIOF}, U_\mathrm{P} \otimes U_\mathrm{I}^{\otimes k} \otimes V_\mathrm{O}^{\otimes k} \otimes V_\mathrm{F}] = 0 \tag{3.24}$$

for every $U, V \in \mathrm{SU}(d)$. We then find the dual SDP problem, and the involved symmetries allowed us to construct a feasible dual point and to show that $F_\mathrm{conj}(d,k) \leq \frac{k+1}{d(d-k)}$, ensuring that the initial construction based on the antisymmetric subspace is optimal.

3.4 Unitary transposition

We now consider the unitary transposition problem, that is, $f(U_d) = U^T$. We start by presenting the optimal probabilistic exact protocol for an arbitrary k and d in the parallel case, result proven in Ref. [13].

Theorem 4 (Ref. [13])**.** *Let* $\widetilde{\widetilde{S}}$ *be a probabilistic* k*-slot parallel superchannel which transforms* k *calls of an arbitrary unitary channel* $\widetilde{U_d}$ *into its transposition, i.e.,*

$$\widetilde{\widetilde{S}}\left(\widetilde{U_d}\right) = p^\mathrm{PAR}_\mathrm{trans}(d,k)\, \widetilde{U_d^T}, \quad \forall U_d \in \mathrm{SU}(d). \tag{3.25}$$

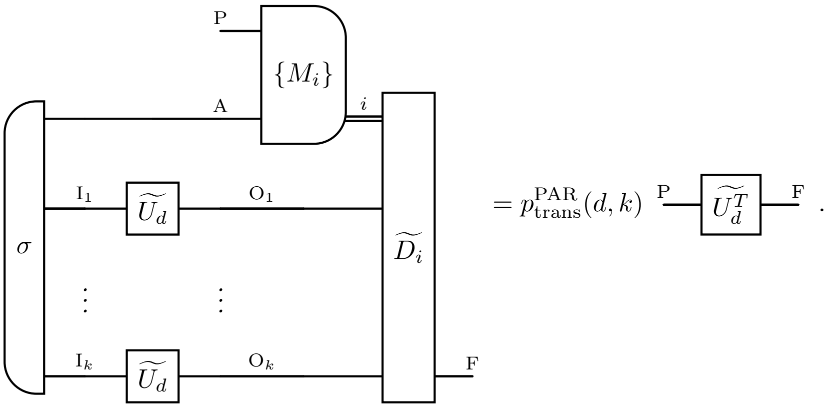
For any $d, k \geq 1$*, the maximal success probability is*

$$p^\mathrm{PAR}_\mathrm{trans}(d,k) = 1 - \frac{d^2-1}{k+d^2-1}. \tag{3.26}$$

This maximal value is attained by a probabilistic port-based teleportation (PBT) protocol [66, 67] or by a unitary storage-and-retrieve (SAR) protocol [25]. That

[3] A vector $|\psi\rangle \in \left(\mathbb{C}^d\right)^{\otimes k}$ belongs to the antisymmetric subspace of $\left(\mathbb{C}^d\right)^{\otimes k}$ if for any permutation $\pi \in S_k$ of a set with k elements, we have $|\psi\rangle_{12\ldots k} = -\,|\psi\rangle_{\pi(12\ldots k)}$ [53].

is, there exist a quantum state σ, a POVM $\{M_i\}_{i=1}^k \cup \{M_F\}$, and a set of quantum channels $\{\widetilde{D_i}\}_{i=1}^k$ such that



(3.27)

Let us now analyse the quantum circuit in Eq. (3.27) that attains optimal probabilistic parallel unitary transposition. First, note that the "target state" that will be used in the system $\mathcal{H}_\mathrm{P}$ is not required at the moment where the k parallel calls to the unitary operations $\widetilde{U_d}$ are done. In Ref. [13], we refer to this subset of parallel superinstruments as delayed input-state parallel superinstruments, and they may be seen as a generalisation of the quantum superinstruments used in unitary storage-and-retrieval [25], and as a probabilistic version of the superchannels used for unitary learning [24].

As mentioned in Thm. 4, optimal parallel unitary transposition can be obtained by a probabilistic PBT protocol as the one described in Ref. [67]. In this case, the decoder operation $\widetilde{D_i}$ is just a partial trace on all subsystems $\mathrm{O}_{\bar{i}}$, where $\bar{i} \neq i$.

Another way to perform optimal parallel unitary transposition is to use the unitary SAR protocol presented in Ref. [25]. For that, we just have to notice that the "encoder state" $\sigma = |\phi\rangle\langle\phi|$ used in Ref. [25] respects the property $U_d^{\otimes k} \otimes \mathbb{1}_d^{\otimes k} |\phi\rangle = \mathbb{1}_d^{\otimes k} \otimes U_d^{T\otimes k} |\phi\rangle$. Hence, we can obtain a unitary transposition simply by performing the k unitary channels $\widetilde{U_d}$ on the complement wires of the encoder state σ. Finally, we note that, while the superinstruments based on PBT and the SAR attain the same success probability of $p_{\text{trans}}^{\text{PAR}}(d,k) = 1 - \frac{d^2-1}{k+d^2-1}$, they consume different resources. As described in Ref. [25], while the PBT approach allows a simpler decoder, the required dimension for the auxiliary space $\mathcal{H}_\mathrm{A}$ for the PBT-based protocol is exponentially greater than the required dimension of the auxiliary space for the SAR-based protocol.

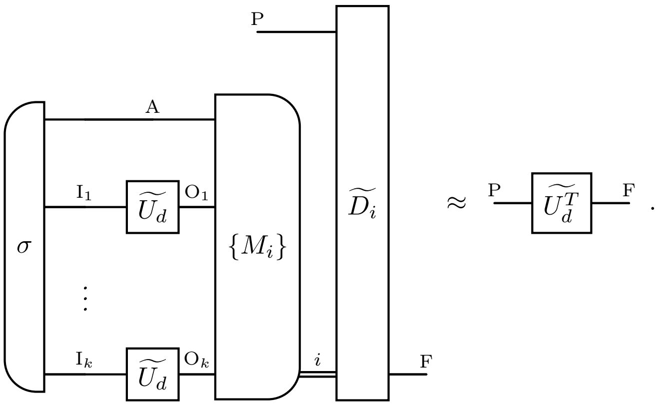
We now focus our attention to deterministic parallel unitary transposition. In Ref. [50], we show that the optimal deterministic approximation for parallel unitary transposition can always be done via a deterministic nonexact version of unitary SAR, also known as quantum unitary learning [24]. Also, as shown in Ref. [24], deterministic unitary SAR can always be done via a unitary estimation protocol. That is, we can first perform k independent calls of the unitary channel $\widetilde{U_d}$ in some quantum state $\sigma \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{A})$ to obtain $\widetilde{U_d}^{\otimes k} \otimes \widetilde{\mathbb{1}}_\mathrm{A}(\sigma)$. Then, we perform a quantum measurement with POVM $\{M_i\}_i$ on the state $\widetilde{U_d}^{\otimes k} \otimes \widetilde{\mathbb{1}}_\mathrm{A}(\sigma)$

and obtain some outcome i, associated to some unitary U'_d. Then, at a later point, we simply prepare the channel $\widetilde{f(U'_d)}$ [see Eq. (3.28)].

Later, in Ref. [51], we showed that any unitary estimation protocol with k calls of the channel $\widetilde{U_d}$ can be converted into deterministic PBT protocol with $N = k - 1$ ports with the same worst-case fidelity. This connection allowed us to relate two seemingly different problems and to understand some of their properties, such as their asymptotic behaviour.

Theorem 5 (Ref. [50, 51])**.** *Let $\widetilde{\widetilde{S}}$ be a deterministic k-slot parallel superchannel which transforms k calls of an arbitrary unitary channel $\widetilde{U_d}$ into its transposition.*

For any dimension d and number of calls k, the optimal deterministic parallel unitary transposition is obtained via an estimation protocol, that is, there exist a quantum state σ, a POVM $\{M_i\}_i$ and a set of quantum channels $\{\widetilde{D_i}\}_i$ such that



(3.28)

For any dimension d and number of calls k, the optimal deterministic parallel unitary transposition respects

$$F^{\text{PAR}}_{\text{trans}}(d,k) = F_{\text{PBT}}(d,k+1), \tag{3.29}$$

where $F_{\text{PBT}}(d,k{+}1)$ is the optimal fidelity for the deterministic PBT protocol [66, 67].

If $k = 1$, the maximal worst-case fidelity over all parallel superchannels $\widetilde{\widetilde{S}}$ is given by

$$F^{\text{PAR}}_{\text{trans}}(d,k=1) = \frac{2}{d^2}. \tag{3.30}$$

If $d = 2$, the maximal worst-case fidelity over all parallel superchannels $\widetilde{\widetilde{S}}$ is given by

$$F^{\text{PAR}}_{\text{trans}}(d=2,k) = 1 - \sin^2\left(\frac{\pi}{k+3}\right) \tag{3.31}$$

If $k \leq d-1$, the maximal worst-case fidelity over all parallel superchannels $\widetilde{\widetilde{S}}$ is given by

$$F^{\text{PAR}}_{\text{trans}}(d,k\leq d-1) = F^{\text{GEN}}_{\text{inv}}(d,k\leq d-1) \tag{3.32}$$

$$= \frac{k+1}{d^2}, \tag{3.33}$$

where $F^{\mathrm{GEN}}_{\mathrm{inv}}(d,k)$ is the maximal worst-case fidelity over all general superchannels $\widetilde{\widetilde{S}}$ for unitary inversion (see Sec. 3.5).

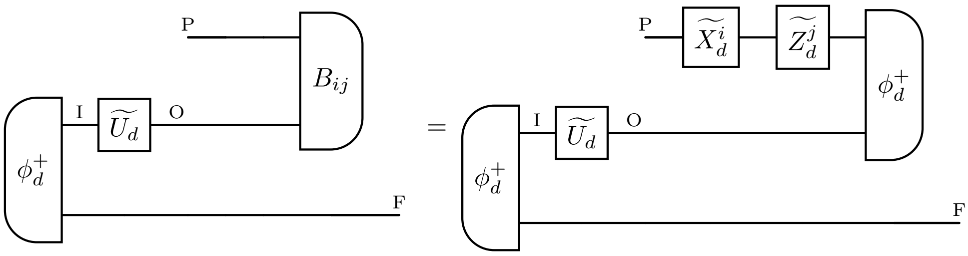
The asymptotic behaviour of $F^{\mathrm{PAR}}_{\mathrm{trans}}(d,k)$ is given by[4]

$$F^{\mathrm{PAR}}_{\mathrm{trans}}(d,k) = 1 - \Theta\left(\frac{d^4}{k^2}\right). \tag{3.37}$$

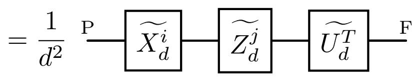
Let us analyse the quantum circuit in Eq. (3.28) that attains optimal deterministic parallel unitary transposition via a unitary estimation protocol. As in the probabilistic case, this is also a delayed-input state protocol. Also, in this one, just after performing the k calls of the unitary channel $\widetilde{U_d}$, we can perform a quantum measurement with POVM $\{M_i\}_i$. In a later moment, when we desire to implement the unitary $\widetilde{U_d^T}$, we just need the classical label i.

We now consider sequential superchannels that transforms k calls of $\widetilde{U_d}$ into its transpose. In Ref. [13], we present an explicit sequential protocol for unitary transposition that is based on the idea of "repeat until success". Before presenting this idea, let us first analyse the optimal probabilistic unitary transposition protocol for the $k=1$ case. The optimal protocol is in very close relation with quantum state teleportation [2], and to gate teleportation [69]. Let $|\phi_d^+\rangle \coloneqq \frac{1}{\sqrt{d}}\, |i\rangle \otimes |i\rangle$ be the maximally entangled state and its associated density matrix be $\phi_d^+ = |\phi_d^+\rangle\langle\phi_d^+|$. The clock operator $Z_d \in \mathcal{L}(\mathbb{C}^d)$ is defined as $Z_d \coloneqq \sum_{i=0}^{d-1} \omega^i\, |i\rangle\langle i|$, where $\omega \coloneqq e^{\frac{2\pi\sqrt{-1}}{d}}$, and the shift operator $X_d \in \mathcal{L}(\mathbb{C}^d)$ is defined as $X_d \coloneqq \sum_{i=0}^{d-1} |i +_d 1\rangle\langle i|$ where the symbol $+_d$ stands for addition modulo d. Note that when $d=2$, the clock and shift operator are the Z and X Pauli operators respectively.

A Bell measurement is a quantum measurement defined with POVM elements $B_{ij} \in \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d)$ given by $B_{ij} \coloneqq Z_d^i X_d^j \otimes \mathbb{1}_d\, |\phi_d^+\rangle\langle\phi_d^+|\, \left(Z_d^i X_d^j \otimes \mathbb{1}_d\right)^\dagger$ where $i,j \in \{0,\ldots,d-1\}$. Direct calculation shows that,



(3.38)



(3.39)

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[4]Here we use the big-O notation, defined as follows [68]:

$$f(x) = O(g(x)) \Leftrightarrow \limsup_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty, \tag{3.34}$$

$$f(x) = \Omega(g(x)) \Leftrightarrow g(x) = O(f(x)), \tag{3.35}$$

$$f(x) = \Theta(g(x)) \Leftrightarrow f(x) = O(g(x)) \text{ and } f(x) = \Omega(g(x)). \tag{3.36}$$

Hence, with probability $p = \frac{1}{d^2}$, we obtain the classical outcomes $i, j \in \{0, \ldots, d-1\}$ and the quantum channel $\widetilde{U_d^T} \circ \widetilde{Z_d^j} \circ \widetilde{X_d^i}$. When the classical outcomes $i = j = 0$ are obtained, we are successful, i.e., $\widetilde{U_d}$ is transformed into $\widetilde{U_d^T}$. But, when $i \geq 0$ or $j \neq 0$, the protocol fails. We could try to do a "correction", that is, to apply some quantum channel in the future linear space $\mathcal{H}_\text{F}$ to recover the possible initial target-state we intend to input at the past space $\mathcal{H}_\text{P}$. That is, we seek a quantum channel $\widetilde{C} : \mathcal{L}(\mathcal{H}_F) \to \mathcal{L}(\mathcal{H}_F)$ such that

$$\text{P} - \boxed{\widetilde{X_d^i}} - \boxed{\widetilde{Z_d^j}} - \boxed{\widetilde{U_d^T}} - \boxed{\widetilde{C}} - \text{F} \quad = \quad \text{P} - \boxed{\widetilde{U_d^T}} - \text{F} \;. \tag{3.40}$$

The obstacle here is that the only channel $\widetilde{C}$ that commutes with any unitary channel $\widetilde{U_d^T}$ is the identity channel. This means that if the map $\widetilde{C}$ is independent of U_d, there is no way to correct the protocol in case of failure.

One idea then is to look for a quantum channel $\widetilde{C}$ that "resets" the protocol in case of failure, that is, to look for a quantum channel $\widetilde{C}$ such that

$$\text{P} - \boxed{\widetilde{X_d^i}} - \boxed{\widetilde{Z_d^j}} - \boxed{\widetilde{U_d^T}} - \boxed{\widetilde{C}} - \text{F} \quad = \quad \text{P} - \boxed{\widetilde{\mathbb{1}_d}} - \text{F} \;. \tag{3.41}$$

Due to essentially the same argument as before, there is no quantum channel $\widetilde{C}$ such that Eq. (3.41) holds for every unitary channel $\widetilde{U_d}$. The trick now is that, since the inverse of U_d^T is U_d^*, if we have access to the channel $\widetilde{U_d^*}$, we can "reset" the protocol, that is,

$$\text{P} - \boxed{\widetilde{X_d^i}} - \boxed{\widetilde{Z_d^j}} - \boxed{\widetilde{U_d^T}} - \boxed{\widetilde{U_d^*}} - \boxed{\widetilde{Z_d^{-j}}} - \boxed{\widetilde{X_d^{-i}}} - \text{F} \quad = \quad \text{P} - \boxed{\widetilde{\mathbb{1}_d}} - \text{F} \;. \tag{3.42}$$

The key idea now is to recall that there exists a deterministic exact superchannel that transforms $k = d-1$ calls of $\widetilde{U_d}$ into $\widetilde{U_d^*}$ [65]. Hence, when failure occurs, we can always "reset" our protocol. This allows us to start it again in a repeat-until-success manner.

One way to summarise this discussion is to say that with $k = d$ calls of $\widetilde{U_d}$ we can have a *success-or-draw* protocol. That is, with probability $p = \frac{1}{d^2}$, we transform $k = d$ calls of an arbitrary $\widetilde{U_d}$ into $\widetilde{U_d^T}$, and with probability $(1-p) = 1 - \left(1 - \frac{1}{d^2}\right)$, we have a "draw", in the sense that we simply apply the identity operator. This allows us to reiterate the $k = d$ protocol to obtain a success probability that grows exponentially in k.

Theorem 6 (Ref. [13, 49])**.** *Let $\widetilde{\widetilde{S}}$ be a probabilistic k-slot sequential superchannel which transforms k calls of an arbitrary unitary channel $\widetilde{U_d}$ into its transposition. That is,*

$$\widetilde{\widetilde{S}}\left(\widetilde{U_d}\right) = p_\text{trans}^\text{SEQ}(d,k)\, \widetilde{U_d^T}, \quad \forall U_d \in \text{SU}(d). \tag{3.43}$$

dimension	number of calls	Parallel	Sequential	General
$d = 2$	$k = 2$	$\frac{2}{5} = 0.4$	$0.4286 \approx \frac{3}{7}$	$0.4444 \approx \frac{4}{9}$
$d = 2$	$k = 3$	$\frac{1}{2} = 0.5$	$0.7500 \approx \frac{3}{4}$	0.9416
$d = 3$	$k = 2$	$\frac{2}{10} = 0.2$	$0.2222 \approx \frac{2}{9}$	$0.2500 \approx \frac{2}{8}$

Table 3.1: Table with optimal success probability for transforming k calls of $\widetilde{U_d}$ into a single call of its transpose $\widetilde{U_d^T}$.

dimension	number of calls	Parallel	Sequential	General
$d = 2$	$k = 2$	$\cos\left(\frac{\pi}{5}\right)^2 = 0.6545$	$0.7500 \approx \frac{3}{7}$	$0.8249 \approx \frac{4}{9}$
$d = 2$	$k = 3$	$\frac{3}{4} = 0.75$	$0.7500 \approx \frac{3}{4}$	0.9416
$d = 3$	$k = 2$	$\frac{1}{3} \approx 0.3333$	0.4072	$\frac{1}{3} \approx 0.4349$

Table 3.2: Table with optimal worst-case fidelity for transforming k calls of $\widetilde{U_d}$ into a single call of its transpose $\widetilde{U_d^T}$.

For any $d, k \geq 1$, the maximal success probability respects[5]

$$p_{\text{trans}}^{\text{SEQ}}(d,k) \geq 1 - \left(1 - \frac{1}{d^2}\right)^{\lceil \frac{k}{d} \rceil}. \tag{3.44}$$

Additionally, the worst-case fidelity when using sequential superchannels respects

$$F_{\text{trans}}^{\text{SEQ}}(d,k) \geq 1 - \left(1 - \frac{1}{d^2}\right)^{\lceil \frac{k}{d} \rceil}. \tag{3.45}$$

Finally, in Refs. [21, 49], we show that the problem of maximising the success probability or the worst-case fidelity for unitary transposition can be solved via SDP. This allows us to numerically solve the problem for small values of d and k. These results are presented in Table 3.1 and Table 3.2. From these tables, we can see that for some values of k and d, general superchannels can outperform sequential ones. In other words, for this task, superchannels without a definite causal order can outperform causally ordered ones for some fixed values of d and k.

3.5 Unitary inversion

We now consider the unitary inversion problem, that is $f(U_d) = U^{-1}$. The possibility of transforming a single call of an arbitrary d-dimensional unitary channel into its inverse was first considered in Ref. [64], where the authors showed that the optimal average fidelity is $F_{\text{inv}}(d, k = 1) = \frac{2}{d^2}$.

[5] Here $\lceil x \rceil$ is the ceiling function of $x \in \mathbb{R}$, that is $\lceil x \rceil$ is the largest integer smaller than or equal to x.

Here we start by presenting the optimal probabilistic exact protocol for an arbitrary k and d in the parallel case, as we proved in Ref. [21].

Theorem 7 (Ref. [21]). *Let $\widetilde{\widetilde{S}}$ be a probabilistic k-slot parallel superchannel which transforms k calls of an arbitrary unitary channel $\widetilde{U_d}$ into its inverse. That is,*

$$\widetilde{\widetilde{S}}\left(\widetilde{U_d}\right) = p_{\text{inv}}^{\text{PAR}}(d,k)\,\widetilde{U_d^{-1}}, \quad \forall U_d \in \text{SU}(d). \tag{3.46}$$

If $k < d-1$, we necessarily have $p_{\text{inv}}(d,k) = 0$.

For any dimension d and number of calls k, the maximal success probability respects[6]

$$1 - \frac{d^2-1}{\lfloor \frac{k}{d-1} \rfloor + d^2 - 1} \leq p_{\text{inv}}^{\text{PAR}}(d,k) \leq 1 - \frac{d^2-1}{k+d^2-1}. \tag{3.47}$$

This maximal value is attained by a delayed-input state superinstrument.

The lower bound in this theorem is established by an explicit construction that we now describe briefly. When the number of accessible calls k is a multiple of $d-1$, we can write $k = k'(d-1)$. In this case, we can obtain the inverse of a unitary operation by transforming the k' groups of $d-1$ unitaries into their complex conjugate using the protocol of Ref. [65]. Then, we can use the k' calls of $\widetilde{U_d^*}$ to implement the optimal probabilistic parallel unitary transposition protocol presented in Thm. 4.

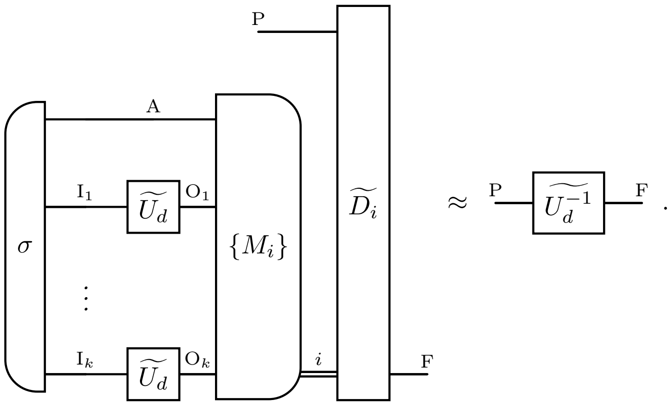
The upper bound is based on the following key observation. Since we know the optimal unitary transposition protocol and that unitary complex conjugation can be done exactly with $k = d-1$ calls, the success probability for the parallel unitary inversion protocol cannot be very high. If it were, we could combine this unitary inversion protocol with high success of probability to the protocol for exact unitary conjugation to obtain a protocol for unitary transposition that is also high.

Theorem 8 (Ref. [49, 51]). *Let $\widetilde{\widetilde{S}}$ be a deterministic k-slot parallel superchannel which transforms k calls of an arbitrary unitary channel $\widetilde{U_d}$ into its inverse.*

For any dimension d and number of calls k, the optimal deterministic parallel unitary transposition is obtained via an estimation protocol, that is, there exist a quantum state σ, a POVM $\{M_i\}$ and a set of quantum channels $\{\widetilde{D_i}\}_i$ such

[6]Here $\lfloor x \rfloor$ is the floor function of $x \in \mathbb{R}$, that is $\lfloor x \rfloor$ is the smallest integer greater than or equals x.

that



(3.48)

For any dimension d and number of calls k, the optimal deterministic parallel unitary transposition respects

$$F_{\text{inv}}^{\text{PAR}}(d,k) = F_{\text{PBT}}(d,k+1), \tag{3.49}$$

where $F_{\text{PBT}}(d,k+1)$ is the optimal fidelity for the deterministic PBT protocol [66, 67].

If $d=2$, the maximal worst-case fidelity over all parallel superchannels $\widetilde{\widetilde{S}}$ is given by

$$F_{\text{inv}}^{\text{PAR}}(d=2,k) = 1 - \sin^2\left(\frac{\pi}{k+3}\right). \tag{3.50}$$

If $k \leq d-1$, the maximal worst-case fidelity over all parallel superchannels $\widetilde{\widetilde{S}}$ is given by

$$F_{\text{inv}}^{\text{PAR}}(d,k\leq d-1) = F_{\text{inv}}^{\text{GEN}}(d,k\leq d-1) \tag{3.51}$$

$$= \frac{k+1}{d^2}, \tag{3.52}$$

where $F_{\text{inv}}^{\text{GEN}}(d,k)$ is the maximal worst-case fidelity over all general superchannels $\widetilde{\widetilde{S}}$.

The asymptotic behaviour of $F_{\text{inv}}^{\text{PAR}}(d,k)$ is given by

$$F_{\text{inv}}^{\text{PAR}}(d,k) = 1 - \Theta\left(\frac{d^4}{k^2}\right). \tag{3.53}$$

We now consider *sequential* superchannels that transform k calls of $\widetilde{U_d}$ into its inverse. Analogously to the unitary transposition case (see the discussion just before Thm. 6), when $k=d$, there exists an instrument that transforms $k=d$ calls of an arbitrary $\widetilde{U_d}$ into $\widetilde{U_d^{-1}}$ with probability $p=\frac{1}{d^2}$. With probability $(1-p) = 1-\left(1-\frac{1}{d^2}\right)$, we have a "draw", in the sense that we simply apply the identity operator. This allows us to reiterate the $k=d$ protocol to obtain a success probability that grows exponentially in k.

Theorem 9 (Ref. [21, 49]). *Let $\widetilde{\widetilde{S}}$ be a probabilistic k-slot sequential superchannel which transforms k calls of an arbitrary unitary channel $\widetilde{U_d}$ into its inverse, that is,*

$$\widetilde{\widetilde{S}}\left(\widetilde{U_d}\right) = p_{\text{inv}}^{\text{SEQ}}(d,k)\,\widetilde{U_d^{-1}}, \quad \forall U_d \in \text{SU}(d). \tag{3.54}$$

For any $d, k \geq 1$, the maximal success probability over all sequential superchannels $\widetilde{\widetilde{S}}$ respects

$$p_{\text{inv}}^{\text{SEQ}}(d,k) \geq 1 - \left(1 - \frac{1}{d^2}\right)^{\lfloor \frac{k+1}{d} \rfloor}. \tag{3.55}$$

Also, the maximal worst-case fidelity when using sequential superchannels respects

$$F_{\text{inv}}^{\text{SEQ}}(d,k) \geq 1 - \left(1 - \frac{1}{d^2}\right)^{\lfloor \frac{k+1}{d} \rfloor}. \tag{3.56}$$

Finally, in Refs. [21, 49], we show that the problem of maximising the success probability or the worst-case fidelity for unitary inversion can be solved via an SDP. This allows us to numerically solve the problem for small values of d and k. These results are presented in Table 3.3 and Table 3.4. From these tables, we can see that for some values of k and d, general superchannels can outperform sequential ones. In other words, for this task, superchannels without a definite causal order can outperform causally ordered ones for some fixed values of d and k.

dimension	number of calls	Parallel	Sequential	General
$d=2$	$k=2$	$\frac{2}{5}=0.4$	$0.4286 \approx \frac{3}{7}$	$0.4444 \approx \frac{4}{9}$
$d=2$	$k=3$	$\frac{1}{2}=0.5$	$0.7500 \approx \frac{3}{4}$	0.9416
$d=3$	$k=2$	$0.1111 \approx \frac{1}{9}$	$0.1111 \approx \frac{1}{9}$	$0.1111 \approx \frac{1}{9}$

Table 3.3: Table with optimal success probability for transforming k calls of $\widetilde{U_d}$ into a single call of its transpose $\widetilde{U_d^{-1}}$.

dimension	number of calls	Parallel	Sequential	General
$d=2$	$k=2$	$\cos\left(\frac{\pi}{5}\right)^2 = 0.6545$	$0.7500 \approx \frac{3}{7}$	$0.8249 \approx \frac{4}{9}$
$d=2$	$k=3$	$\frac{3}{4}=0.75$	$0.7500 \approx \frac{3}{4}$	0.9416
$d=3$	$k=2$	$\frac{1}{3} \approx 0.3333$	$\frac{1}{3} \approx 0.3333$	$\frac{1}{3} \approx 0.3333$

Table 3.4: Table with optimal worst-case fidelity for transforming k calls of $\widetilde{U_d}$ into a single call of its transpose $\widetilde{U_d^{-1}}$.

3.6 The existence of success-or-draw protocols

In order to construct a sequential superinstrument for unitary transposition and unitary inversion, we utilised the concept of success-or-draw protocols. That is, we started with a one-slot "success-or-failure" superinstrument that transforms a single call of $\widetilde{U_d}$ into some channel $\widetilde{f(U_d)}$ with probability p and, with probability $1-p$, we fail and obtain some other quantum channel $\widetilde{F}$. Since the failure channel $\widetilde{F}$ is not the identity channel, we cannot reiterate the protocol in a repeat-until-success manner.

In Ref. [48] we prove that the method used for unitary transposition and unitary inversion is actually general. That is, if there exists a "success-or-failure" one-slot superinstrument that transforms a single call of $\widetilde{U_d}$ into some channel $\widetilde{f(U_d)}$ with probability p_{U_d}, then there exist an $\epsilon > 0$ and a success-or-draw $(k = d)$-slot sequential superchannel that transforms $k = d$ calls of $\widetilde{U_d}$ into $\widetilde{f(U_d)}$ with some probability $\epsilon\, p_{U_d}$, and with probability $1 - \epsilon\, p_{U_d}$, we obtain the identity channel $\widetilde{\mathbb{1}_d}$.

Theorem 10 (Ref. [48])**.** *Let $\widetilde{\widetilde{S}}$ be a probabilistic 1-slot parallel superchannel which transforms a single of an arbitrary unitary channel $\widetilde{U_d}$ into $\widetilde{f(U_d)}$ for some arbitrary $f : \mathrm{SU}(d) \to \mathrm{SU}(d')$ with probability p, that is*

$$\widetilde{\widetilde{S}}\left(\widetilde{U_d}\right) = p\,\widetilde{f(U_d)}, \quad \forall U_d \in \mathrm{SU}(d). \tag{3.57}$$

Then, there exists a $(k = d)$-slot sequential superinstrument $\{\widetilde{\widetilde{S'}}, \widetilde{\widetilde{F}}\}$ and some $\epsilon > 0$ such that

$$\widetilde{\widetilde{S'}}\left(\widetilde{U_d}^{\otimes d}\right) = \epsilon\, p_{U_d}\,\widetilde{f(U_d)}, \quad \forall U_d \in \mathrm{SU}(d) \tag{3.58}$$

$$\widetilde{\widetilde{F}}\left(\widetilde{U_d}^{\otimes d}\right) = (1 - \epsilon\, p_{U_d})\,\widetilde{\mathbb{1}_d}, \quad \forall U_d \in \mathrm{SU}(d). \tag{3.59}$$

3.7 Outlook

In this chapter, we analysed the problem of transforming k calls of a unitary channel $\widetilde{U_d}$ into $\widetilde{f(U_d)}$, where f can be the complex conjugate, transposition, or inversion. In the literature, other transformations between unitary operations have also been considered. For instance, Refs. [60, 70] analyse the problem of unitary cloning $f(U_d) = U^n$ where $n > 1$. Ref. [71] considers the problem of unitary iteration, i.e., $f(U_d) = U_d^n$, where $n \in \mathbb{N}$ and U_d^n stands for composing the operator U_d with itself n times. Another very relevant transformation that is widely studied is the case of unitary controlisation [18–20], i.e., $f(U_d) = |0\rangle\langle 0| \otimes \mathbb{1}_d + e^{\theta_{U_d}} |1\rangle\langle 1| \otimes U_d$, where $\theta_{U_d} \in \mathbb{R}$.

Some works also consider scenarios where only a restricted set of superchannels is available. One relevant case is a scenario where one does not have access to an auxiliary system, case also referred to it as Markovian processes [45]. In such scenarios, implementing functions like $f(U_d) = \mathbb{1}_d$ is a nontrivial relevant task [72]. Also, Refs. [73, 74] consider the problem of inverting unitary operations on a restricted class of superchannels where one can only perform encoders and decoders on the auxiliary systems. Some other works consider the problem where the unitary has a Hamiltonian dynamics structure [75], that is, U_d^{itH}, where the self-adjoint operator H is unknown, but we have control of the time parameter $t \in \mathbb{R}$.

Before finishing this chapter, we would like to state the results of Refs. [76, 77]—that appeared some years after Thm. 6 and Thm. 9—which presents an explicit sequential superchannel that attains unitary inversion and unitary transposition with a success probability that grows exponentially with the number of calls k. In Ref. [76] the authors show that, for the qubit case, i.e., $d = 2$, there exist a deterministic and exact sequential 4-slot superchannel such that $\widetilde{\widetilde{S}}\left(\widetilde{U_2}^{\otimes 4}\right) = \widetilde{U_2^{-1}}$. Then, in Ref. [77], it was shown that for any dimension d, there exists a finite number of calls $k = \mathcal{O}(d^2)$ and a sequential k-slot superchannel $\widetilde{\widetilde{S}}$ such that $\widetilde{\widetilde{S}}\left(\widetilde{U_d}^{\otimes k}\right) = \widetilde{U_d^{-1}}$. In other words, in a sequential scenario, unitary inversion (and also unitary transposition) can be done deterministically and exactly for some finite number of calls k.

Despite all recent progress, several open questions and open directions remain about transformations between unitary channels. For example, how does the situation change if we know the input state that will be applied in the past space $\mathcal{H}_\text{P}$? How does the situation change when some particular knowledge of the input channel $\widetilde{U_d}$ is available. For instance, if we are given a single call of an arbitrary channel $\widetilde{U_d}$ and a quantum state $|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is an eigenvector of U_d, it is known that we can implement the unitary controlisation function $f(U_d) = |0\rangle\langle 0| \otimes \mathbb{1}_d + e^{\theta_{U_d}} |1\rangle\langle 1| \otimes U_d$. This can be contrasted with the fact that, without extra resources, universal unitary controlisation is impossibility, even probabilistically, for any finite number of calls k [20].

Chapter 4

Discriminating quantum channels

This chapter contains original results extracted from the following publications:

- [12] Strict Hierarchy between Parallel, Sequential, and Indefinite-Causal-Order Strategies for Channel Discrimination
  J. Bavaresco, M. Murao, **M.T. Quintino**
  Phys. Rev. Lett. 127, 200504 (2021)
  arXiv:2011.08300 [quant-ph]

- [78] Unitary channel discrimination beyond group structures: Advantages of sequential and indefinite-causal-order strategies
  J. Bavaresco, M. Murao, **M.T. Quintino**
  J. Math. Phys. 63, 042203 (2022)
  arXiv:2105.13369 [quant-ph]

- [79] The quantum switch is uniquely defined by its action on unitary operations
  Q. Dong, **M.T. Quintino**, A. Soeda, M. Murao
  Quantum 7, 1169 (2023)
  arXiv:2106.00034 [quant-ph]

4.1 The quantum channel discrimination problem and quantum testers

4.1.1 Introduction

The discrimination of quantum operations is one of the most fundamental tasks in quantum information science. It relates to the elementary ability to experimentally distinguish among different quantum dynamics, which comes into play, for example, in tasks associated to certification of quantum circuits.

A plethora of interesting results on this topic has been demonstrated over the course of the years. For the scenario in which the task consists of the discrimination of a pair of channels using only one query, or copy, of an unknown channel, the problem of finding the maximal probability of successful discrimination has been related to the Helstrom measurement [80] and the diamond norm [81, 82]. In striking contrast with the problem of state discrimination – in which any two states can only be perfectly discriminated with a finite number of copies if they are orthogonal – it has been shown that any pair of unitary channels can always be perfectly discriminated for some finite number of copies [83]. Still concerning pairs of unitary channels, it has been shown that there is no advantage of sequential strategies over parallel strategies for discrimination with any finite number of copies [11]. However, for general channels, there can be an advantage of sequential strategies over parallel ones, as demonstrated in Ref. [84] with an example of two qubit-ququart entanglement breaking channels that can be perfectly discriminated with a sequential strategy but not with a parallel one. Additionally, during the writing process of our article Ref. [12], Ref. [85] was uploaded to arXiv with an example of two qubit-qubit generalized amplitude damping channels.

In a related task that consists of the discrimination of two "no-signalling bipartite channels", a more general strategy was constructed from the quantum switch [86]. This strategy involved indefinite causal order, and it was shown to not only provide an advantage over causal, i.e. sequential and parallel, strategies, but also to allow for perfect discrimination, which would otherwise not be achievable [14]. This phenomenon already hints that indefinite causal order could be useful for the task of channel discrimination, similarly to how it has proven to be advantageous for other tasks, such as the inversion of unknown unitary operations [13] and quantum computation [87].

Additionally, we analyse a set of general strategies based on the quantum switch [27], a two-slot general superchannel that is not a two-slot sequential superchannel. In order to understand the quantum switch better, we show that the quantum switch is uniquely defined by its action on unitary operators.

In this chapter, we cover the results from originally presented in Refs. [12, 78, 79].

4.1.2 The single-call channel discrimination problem

The task of minimum-error channel discrimination works as follows: With probability p_i, Alice is given an unknown quantum channel $\widetilde{C_i} : \mathcal{L}(\mathcal{H}_\mathrm{I}) \to \mathcal{L}(\mathcal{H}_\mathrm{O})$, drawn from an ensemble $\mathcal{E} = \{p_j, \widetilde{C_j}\}_{j=1}^N$ that is known to her. Being allowed to use k calls of the given channel $\widetilde{C_i}$, her task is to determine which channel she received, by performing operations on this channel and guessing the value of $i \in \{1, \ldots, N\}$. This problem is equivalent to Alice extracting the "classical information" i which is encoded in the channel $\widetilde{C_i}$. In the simplest case of this task, when Alice is allowed to use $k = 1$ call of the channel she received, the most general quantum operations that Alice could apply in her laboratory are to send part of a potentially entangled state $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{I} \otimes \mathcal{H}_\mathrm{A})$ through the channel $\widetilde{C_i}$, and jointly measure the output with a positive operator-valued measure (POVM) $\{M_a\}, M_a \in \mathcal{L}(\mathcal{H}_\mathrm{O} \otimes \mathcal{H}_\mathrm{A})$, announcing the outcome of her measurement as her guess. Then, her probability of correctly guessing the value of i is given by

$$p_\mathrm{succ} \coloneqq \sum_{i=1}^N p_i \operatorname{tr}\left[(\widetilde{C_i} \otimes \widetilde{\mathbb{1}})(\rho)\, M_i\right], \tag{4.1}$$

where $\widetilde{\mathbb{1}}$ is the identity map on $\mathcal{L}(\mathcal{H}_\text{A})$. The maximal probability of success in this case is then given by

$$p^*_\text{succ} := \max_{\{\rho,\{M_a\}\}} p_\text{succ} \tag{4.2}$$
$$= \max_{\{\rho,\{M_a\}\}} \sum_{i=1}^{N} p_i \operatorname{tr}\left[(\widetilde{C_i} \otimes \widetilde{\mathbb{1}})(\rho)\, M_i\right], \tag{4.3}$$

where the optimisation runs over all possible strategies using quantum states ρ and quantum POVMs $\{M_a\}_{a=1}^N$

As discussed in Sec. 2.3.4, the tester formalism allows for a simpler characterisation of Alice's strategies, who can now optimize over general testers $\{T_i\}_i$ to achieve a maximal probability of successful discrimination. That is, we can write the maximal probability of success of Eq. (4.2) as

$$p^*_\text{succ} = \max_{\{T_i\}_i} \sum_{i=1}^{N} p_i \operatorname{tr}(T_i\, C_i), \tag{4.4}$$

there the optimisation runs over all possible testers.

Now let us analyse the more interesting case in which Alice receives two calls of the channel C_i. With two calls, Alice has the freedom of choosing how to concatenate these channels in order to gain more information about them.

4.1.3 Channel discrimination with two parallel calls

The first and simplest option is to apply the two calls of the unknown channel in parallel, by sending a joint state $\rho \in \mathcal{L}(\mathcal{H}_{\text{I}_1} \otimes \mathcal{H}_{\text{I}_2} \otimes \mathcal{H}_\text{A})$ through both calls of C_i and then measuring the output with a POVM $M := \{M_i\}, M_i \in \mathcal{L}(\mathcal{H}_{\text{O}_1} \otimes \mathcal{H}_{\text{O}_2} \otimes \mathcal{H}_\text{A})$, where $\mathcal{H}_{\text{I}_1}(\mathcal{H}_{\text{I}_2})$ represents the input space of the first (second) call of C_i, and equivalently for the output spaces. Just like in the single slot case, a set of operators $T^\text{PAR} := \{T_i^\text{PAR}\}$ is a parallel tester if and only if

$$T_i \geq 0 \tag{4.5}$$
$$\sum_i T_i = \sigma_{\text{I}_1\text{I}_2} \otimes \mathbb{1}_{\text{O}_1\text{O}_2} \tag{4.6}$$
$$\operatorname{tr}(\sigma_{\text{I}_1\text{I}_2}) = 1. \tag{4.7}$$

Also, all parallel testers have a quantum realisation in terms of states and measurements, that is, there exist a state $\rho \in \mathcal{L}(\mathcal{H}_{\text{I}_1} \otimes \mathcal{H}_{\text{I}_2} \otimes \mathcal{H}_\text{A})$ and a POVM $\{M_i\}_i$, $M_i \in \mathcal{L}(\mathcal{H}_\text{A} \otimes \mathcal{H}_{\text{O}_1} \otimes \mathcal{H}_{\text{O}_2})$ such that $T_i^\text{PAR} = \rho * M_i^T$, see Fig. 4.1(a).

4.1.4 Channel discrimination with two sequential calls

More generally, Alice could use her two calls of C_i sequentially, first sending a state $\rho \in \mathcal{L}(\mathcal{H}_{\text{I}_1} \otimes \mathcal{H}_{\text{A}_1})$ through the first call of C_i, next applying to the output a general channel $\widetilde{E} : \mathcal{L}(\mathcal{H}_{\text{O}_1} \otimes \mathcal{H}_{\text{A}_1}) \to \mathcal{L}(\mathcal{H}_{\text{I}_2} \otimes \mathcal{H}_{\text{A}_2})$, then sending part of the output of channel $\widetilde{E}$ through the second call of C_i and finally measuring the output with a POVM $\{M_i\}_i, M_i \in \mathcal{L}(\mathcal{H}_{\text{O}_2} \otimes \mathcal{H}_{\text{A}_2})$. As in the sequential superchannel scenario – a sequential tester $T^\text{SEQ} := \{T_i^\text{SEQ}\}$ can always be expressed as $T_i^{\text{SEQ}} = \rho * E * M_i^T$, where $E \in \mathcal{L}(\mathcal{H}_{\text{O}_1} \otimes \mathcal{H}_{\text{A}_1} \otimes \mathcal{H}_{\text{I}_2} \otimes \mathcal{H}_{\text{A}_2})$ is the Choi operator of map $\widetilde{E}$, meaning it can always be realised by quantum circuit [8] (see Fig. 4.1(b)). In fact, a quantum sequential tester can be seen as a sequential superinstrument where the past space $\mathcal{H}_P$ and future space $\mathcal{H}_F$ are one-dimensional, that is $\mathcal{H}_P \cong \mathcal{H}_\text{F} \cong \mathbb{C}$. From this reasoning, it follows that $\{T_i\}_i$ is a sequential tester if and only if the operators $T_i \in \mathcal{L}(\mathcal{H}_{\text{I}_1} \otimes \mathcal{H}_{\text{O}_1} \otimes \mathcal{H}_{\text{I}_2} \otimes \mathcal{H}_{\text{O}_2})$ respect

$$T_i \geq 0 \tag{4.8}$$

$$W \coloneqq \sum_i T_i \tag{4.9}$$

$$W = {}_{\text{O}_2}W \tag{4.10}$$

$${}_{\text{I}_2\text{O}_2}W = {}_{\text{O}_1\text{I}_2\text{O}_2}W \tag{4.11}$$

$$\text{tr}(W) = d_{\text{O}_1} d_{\text{O}_2}. \tag{4.12}$$

4.1.5 Channel discrimination with two calls without a definite causal order and the quantum switch

Parallel and sequential strategies have long been regarded as the most general strategies for channel discrimination. However, there is a more general strategy for channel discrimination than the sequential one, which arises from the following reasoning: we define *general two-slot tester* as the most general set of operators $T^{\text{GEN}} = \{T_i^{\text{GEN}}\}$ that map a pair of quantum channels, represented by their Choi operators $C_A \in \mathcal{L}(\mathcal{H}_{\text{I}_1} \otimes \mathcal{H}_{\text{O}_1})$ and $C_B \in \mathcal{L}(\mathcal{H}_{\text{I}_2} \otimes \mathcal{H}_{\text{O}_2})$, to a valid probability distribution according to $p(i|C_A, C_B) = \text{tr}\big[(C_A \otimes C_B) T_i^{\text{GEN}}\big]$. Note that, as in the sequential case, a general tester is equivalent to a quantum superinstrument in which $\mathcal{H}_P \cong \mathcal{H}_\text{F} \cong \mathbb{C}$. This is also equivalent to saying that all tester elements are positive semidefinite, i.e., $T_i \geq 0$, and that $W \coloneqq \sum_i T_i$ is a bipartite process matrix [28, 34], where a bipartite process matrix is the Choi operation of a two-slot superchannel where $\mathcal{H}_\text{P} \cong \mathcal{H}_\text{F} \cong \mathbb{C}$.

There is a special particular case of general testers that deserves special attention, which is the quantum testers that can be realised with the aid of the *quantum switch* [27]. The quantum switch is a two-slot general superchannel

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_{\text{I}_1} \otimes \mathcal{H}_{\text{I}_2}) \to \mathcal{L}(\mathcal{H}_{\text{O}_1} \otimes \mathcal{H}_{\text{O}_2})] \to [\mathcal{L}(\mathcal{H}_\text{P}) \to \mathcal{L}(\mathcal{H}_\text{F})] \tag{4.13}$$

where the past and future spaces respectively decompose as $\mathcal{H}_\text{P} = \mathcal{H}_{\text{P}_c} \otimes \mathcal{H}_{\text{P}_t}$, and $\mathcal{H}_\text{F} = \mathcal{H}_{\text{F}_c} \otimes \mathcal{H}_{\text{F}_t}$, where $\mathcal{H}_{\text{P}_c} \cong \mathcal{H}_{\text{F}_c} \cong \mathbb{C}^2$. Here, the subindexes c and t respectively correspond to "control" and "target", for reasons that should be clear soon. For any pair of unitary operators $U_A, U_B \in \text{SU}(d)$, the quantum switch superchannel $\widetilde{\widetilde{S}}$ behaves as

$$\widetilde{\widetilde{S}}\left(\widetilde{U_A} \otimes \widetilde{U_B}\right) = \widetilde{V} \tag{4.14}$$

where

$$V \coloneqq |0\rangle\langle 0|_c \otimes U_B U_A + |1\rangle\langle 1|_c \otimes U_A U_B. \tag{4.15}$$

The operator V can be interpreted as a quantum control of the order with which we compose the operations U_A and U_B. That is, for any "target state"

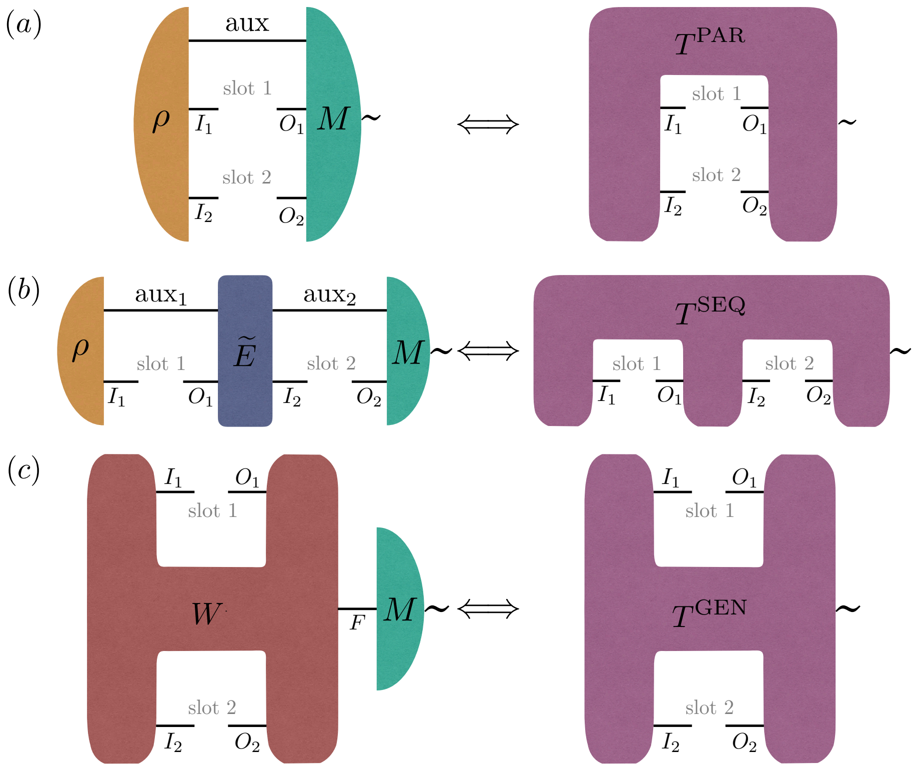


Figure 4.1: Schematic representation of the realisation of every two-call (a) parallel tester T^{PAR} with a state ρ and a POVM M, (b) sequential tester T^{SEQ} with a state ρ, a channel $\widetilde{E}$, and a POVM M, and (c) general tester T^{GEN} with a process matrix W and a POVM M.

$|\psi\rangle \in \mathcal{H}_{\mathrm{P}_t}$ it holds that

$$V\,|0\rangle_c \otimes |\psi\rangle = |0\rangle \otimes (U_B U_A\,|\psi\rangle) \tag{4.16}$$

$$V\,|1\rangle_c \otimes |\psi\rangle = |1\rangle \otimes (U_A U_B\,|\psi\rangle)\,. \tag{4.17}$$

Note that, since we can set the quantum control state as to $|+\rangle := \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, the resulting transformation can be viewed as a superposition of two different causal orders, according to

$$V\,|+\rangle_c \otimes |\psi\rangle = \frac{1}{\sqrt{2}}\Big[\,|0\rangle \otimes (U_B U_A\,|\psi\rangle) + |1\rangle \otimes (U_A U_B\,|\psi\rangle)\Big]. \tag{4.18}$$

Before proceeding, let us note that in the definition of the quantum switch presented above, we do not specify the action of the quantum switch supermap on non-unitary channels. Or, the action of the quantum switch supermap on part of bipartite channels. In principle, this may lead to some ambiguity, in the sense that, there may be different two-slot superchannels that act as the quantum switch on unitary operators, but have a different action on non-unitary channels.

In Ref. [79] we prove that there is no ambiguity by defining the quantum switch only on unitary operation.

Theorem 11 ([79]). *There exists a unique completely CP preserving supermap*

$$\widetilde{\widetilde{S}} : [\mathcal{L}(\mathcal{H}_{\mathrm{I}_1} \otimes \mathcal{H}_{\mathrm{I}_2}) \to \mathcal{L}(\mathcal{H}_{\mathrm{O}_1} \otimes \mathcal{H}_{\mathrm{O}_2})] \to [\mathcal{L}(\mathcal{H}_{\mathrm{P}_c} \otimes \mathcal{H}_{\mathrm{P}_t}) \to \mathcal{L}(\mathcal{H}_{\mathrm{P}_c} \otimes \mathcal{H}_{\mathrm{P}_t})] \tag{4.19}$$

respecting

$$\widetilde{\widetilde{S}}\left(\widetilde{U_A} \otimes \widetilde{U_B}\right) = \widetilde{V}, \qquad \forall U_A, U_B \in \mathrm{SU}(d) \tag{4.20}$$

where $V := |0\rangle\langle 0|_c \otimes U_B U_A + |1\rangle\langle 1|_c \otimes U_A U_B$.

Moreover, this unique completely CP preserving supermap $\widetilde{\widetilde{S}}$ *is also TP preserving, thus a two-slot quantum superchannel.*

We now consider general two-slot quantum testers that can be constructed by (i) "plugging" an arbitrary quantum state $\rho \in \mathcal{L}(\mathcal{H}_\mathrm{P})$ in the quantum switch, (ii) performing arbitrary unitary operations before or after any input operation, and (iii) performing an arbitrary quantum measurement on the future space $\mathcal{H}_\mathrm{F}$. This class of strategies can also be viewed in terms of *quantum switchlike superchannels*, a class of general superchannels that we discuss in appendix E of [78]. When considering general superchannels of two slots, this class of switchlike strategies coincides to what we define as separable testers in Ref. [78]. The proof of this equivalence is done in Ref. [46]. We say that a general two-slot tester $\{T_i\}_i$ is a separable tester when $W = \sum_i T_i$ is a causally separable bipartite process matrix [28, 34]. Where, a bipartite process matrix is causally separable if it can be written as a convex combination of ordered process matrices. And, an ordered process matrix is the Choi operator of a sequential superchannel where $\mathcal{H}_\mathrm{P} \cong \mathcal{H}_\mathrm{F} \cong \mathbb{C}$. Notice that, when considering the $(k = 2)$-slot case, there are two different orders for sequential superchannel, one order where the first slot is used before the second, and another order where the second slot is used before the first.

We have thus defined another set of strategies, the separable testers, that are equivalent to switchlike testers. This set of strategies contains sequential testers, and it is contained in the set of general testers.

4.2 Rigorous upper and lower bounds via computer assisted proofs

With our constructed unified framework for channel discrimination at hand, we can now define the maximum probability of successful discrimination under each of the four described strategies by allowing Alice to optimize over different classes of testers. The maximum probability of successful discrimination of a channel ensemble $\mathcal{E} = \{p_i, C_i\}$ using two copies under strategy $\mathcal{S} \in \{\mathrm{PAR}, \mathrm{SEQ}, \mathrm{SEP}, \mathrm{GEN}\}$ then reads

$$P^{\mathcal{S}} := \max_{\{T^{\mathcal{S}}\}} \sum_{i=1}^{N} p_i \operatorname{tr}\left(T_i^{\mathcal{S}}\, C_i^{\otimes 2}\right). \tag{4.21}$$

It is clear that these four strategies – parallel, sequential, separable, and general – form a hierarchy since the set of testers that they define is a superset of the previous one, in this exact order, implying the relation $P^{\text{PAR}} \leq P^{\text{SEQ}} \leq P^{\text{SEP}} \leq P^{\text{GEN}}$ for any fixed ensemble. We show that, in fact, all these three inequalities can be strictly satisfied by explicitly calculating all $P^{\mathcal{S}}$ for a specific ensemble.

To compute the values of $P^{\mathcal{S}}$, we phrase the optimisation problems that define it in terms of an SDP. Essentially,

$$\begin{aligned} \textbf{given} \quad & \{p_i, C_i\} \\ \textbf{maximize} \quad & \sum_i p_i \operatorname{tr}\left(T_i^{\mathcal{S}}\, C_i^{\otimes 2}\right) \\ \textbf{subject to} \quad & \{T_i^{\mathcal{S}}\} \text{ is a tester with strategy } \mathcal{S}. \end{aligned} \tag{4.22}$$

As we detailed in Ref. [12], the SDP presented in Eq. (4.22) can be equivalently solved by its dual problem

$$\begin{aligned} \textbf{given} \quad & \{p_i, C_i\} \\ \textbf{minimize} \quad & \lambda \\ \textbf{subject to} \quad & p_i\, C_i^{\otimes 2} \leq \lambda\, \overline{W}^{\mathcal{S}} \quad \forall\, i, \end{aligned} \tag{4.23}$$

where $\overline{W}^{\mathcal{S}}$ lies in the dual affine[1] of the set of processes $\mathcal{W}^{\mathcal{S}}$.

SDPs can be solved by efficient numerical packages which, despite being in practice accurate, suffer from imprecisions that come from the use of floating-point variables. In order to overcome this issue, in Ref. [12], we developed an algorithm for computer-assisted proofs (see [88, 89]). Using our computer-assisted proof method, which does not make use of floating-point variables, we obtain exact upper and lower bounds for $P^{\mathcal{S}}$, arriving at a result that has the same mathematical rigour as an analytical proof.

4.3 Strict hierarchy between parallel, sequential, and indefinite-causal-order strategies

We now state the main theorem presented in Ref. [12]. The proof of this result makes use of the computer-assisted proof methods, and all code used in this work is publicly available at Ref. [90].

Theorem 12 (Ref. [12])**.** *In the simplest instance of a channel discrimination task using $k = 2$ calls, i.e., discrimination between $N = 2$ qubit-qubit channels, there exist ensembles for which the maximal probability of successful discrimination of parallel, sequential, separable, and general strategies obey the strict hierarchy*

$$P^{PAR} < P^{SEQ} < P^{SEP} < P^{GEN}. \tag{4.24}$$

In particular, for the task of discriminating the channel ensemble given by $p_1 = p_2 = \frac{1}{2}$, an amplitude damping channel[2] with damping parameter $\gamma_1 = 0.37$

[1] Let $\mathcal{W} \subseteq \mathcal{L}(\mathcal{H})$ be a set of linear operators. A linear operator $\overline{W} \in \mathcal{L}(\mathcal{H})$ is an element of the dual affine set $\overline{\mathcal{W}}$ when $\operatorname{tr}\left(\overline{W}^{\dagger} W\right) = 1$ for all $W \in \mathcal{W}$ [64].

[2] The action of an amplitude damping channel on a qubit state is given by $\widetilde{A}_{\gamma}(\rho) = K_0 \rho K_0^{\dagger} + K_1 \rho K_1^{\dagger}$, where $K_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}\, |1\rangle\langle 1|$ and $K_1 = \sqrt{\gamma}\, |0\rangle\langle 1|$.

and another amplitude damping parameter $\gamma_2 = 0.87$, we have

$$\frac{8101}{10000} < P^{PAR} < \frac{8102}{10000} < \frac{8161}{10000} < P^{SEQ} < \frac{8162}{10000} < \quad (4.25)$$

$$\frac{8166}{10000} < P^{SEP} < \frac{81665}{100000} < \frac{8167}{10000} < P^{GEN} < \frac{8168}{10000}. \quad (4.26)$$

4.4 Unitary quantum channel discrimination

As discussed in the preliminary Chapter 2, unitary channels are precisely the quantum channels whose inverse map is also a quantum channel. For this reason, unitary channels are a very important subset of quantum channels. Since unitary quantum channels always have an inverse, some sets of unitary channels can have a group structure. We say that a set of unitary channels $\{\widetilde{U_i}\}_i$ form a group if

- $\widetilde{\mathbb{1}} \in \{\widetilde{U_i}\}_i$
- If $\widetilde{U} \in \{\widetilde{U_i}\}_i$, then $\widetilde{U^{-1}} \in \{\widetilde{U_i}\}_i$
- If $\widetilde{U_A}, \widetilde{U_B} \in \{\widetilde{U_i}\}_i$, then $\widetilde{U_B} \circ \widetilde{U_A} \in \{\widetilde{U_i}\}_i$.

The task of discriminating unitary channel that form a group and are uniformly distributed can be tackled by group representation theory methods, what often greatly simplify the problem, see e.g., cite [55, 59, 86] In Ref. [11], it was shown that for an ensemble composed of a set of unitary channels that forms a group and a uniform distribution, parallel strategies do not only perform as well as sequential ones. Then, in Ref. [78] we extend this result to general strategies, as detailed in the theorem below.

Theorem 13 (Ref. [78])**.** *For ensembles composed of a uniform probability distribution and a set of unitary channels that forms a group up to a global phase, in discrimination tasks that allow for k copies, parallel strategies are optimal, even when considering general strategies.*

More specifically, let $\mathcal{E} = \{p_i, \widetilde{U_i}\}_i$ be an ensemble with N unitary channels where $p_i = \frac{1}{N}\ \forall\, i$ and the set of channels $\{\widetilde{U_i}\}_i$ forms a group. Then, for any number of copies k, and for every general tester $\{T_i^{GEN}\}$, there exists a parallel tester $\{T_i^{PAR}\}_i$, such that

$$\frac{1}{N}\sum_{i=1}^{N} \operatorname{tr}\left(T_i^{PAR}\, |U_i\rangle\!\rangle\langle\!\langle U_i|^{\otimes k}\right) = \frac{1}{N}\sum_{i=1}^{N} \operatorname{tr}\left(T_i^{GEN}\, |U_i\rangle\!\rangle\langle\!\langle U_i|^{\otimes k}\right). \quad (4.27)$$

However, in Ref. [78] we show that when the unitaries do not form a group, sequential strategies can outperform parallel ones, and general strategies can outperform sequential ones. Moreover, if we have a set of unitaries that form a group, but they are distributed in a non-uniform way, it is also the case that sequential strategies can outperform parallel ones, and general strategies can outperform sequential ones.

Theorem 14 (Ref. [78])**.** *There exist ensembles of unitary channels for which sequential strategies of discrimination outperform parallel strategies. Moreover,* *sequential strategies can achieve perfect discrimination in some scenarios where the maximal probability of success of parallel strategies is strictly less than one.*

Additionally, there exist ensembles of unitary channels for which general strategies of discrimination outperform sequential strategies.

The proof of Thm. 14 is done by presenting explicit examples. The proof that these examples are indeed correct is provided in [78] and applies the method of computer-assisted proofs developed in Ref. [12].

Let us start with the case where the set of unitary channels does not form a group, but the probability distribution of the ensemble is uniform. In the following, σ_x, σ_y, and σ_z are the Pauli operators and $H := |+\rangle\langle 0| + |-\rangle\langle 1|$, where $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, is the Hadamard gate.

Example 1. *The ensemble composed by a uniform probability distribution and $N = 4$ qubit unitary channels given by $\{U_i\} = \{\mathbb{1}, \sqrt{\sigma_x}, \sqrt{\sigma_y}, \sqrt{\sigma_z}\}$, in a discrimination task that allows for $k = 2$ copies, can be discriminated under a sequential strategy with probability of success $P^{SEQ} = 1$ while any parallel strategy yields $P^{PAR} < 1$.*

A straightforward sequential strategy that attains perfect discrimination of this ensemble can be constructed by first noticing that $\sqrt{\sigma_i}\sqrt{\sigma_i} = \sigma_i$, hence, a simple composition of the unitary operators U_i leads to the Pauli operators, which are perfectly discriminated with a bipartite maximally entangled state and a joint measurement in the Bell basis. Here we note, after the publicatin of Ref. [78], we found an alternative proof that $P^{\text{PAR}} < 1$ that is not based on SDP and does not make use of computer assisted methods [91].

Another example of this phenomenon is showcased by the ensemble $\{U_i\} = \{\mathbb{1}, \sigma_x, \sigma_y, \sqrt{\sigma_z}\}$ with uniform probability distribution, which also satisfies $P^{\text{PAR}} < P^{\text{SEQ}} = 1$ for a discrimination task with $k = 2$ copies. In Ref. [78], we show that such an ensemble can actually be discriminated perfectly by a sequential strategy that uses, on average, 1.5 copies.

The next example concerns a set of unitary channels that forms a group, but the probability distribution of the ensemble is not uniform.

Example 2. *Let $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z, H, \sigma_x H, \sigma_y H, \sigma_z H\} = \{U_i\}$ be a tuple of $N = 8$ unitary channels that forms a group up to a global phase, and let $\{p_i\}$ be a probability distribution in which each element p_i is proportional to the i-th digit of the number $\pi \approx 3.1415926$, that is, $\{p_i\} = \{\frac{3}{31}, \frac{1}{31}, \frac{4}{31}, \ldots, \frac{6}{31}\}$. For the ensemble $\{p_i, U_i\}$, in a discrimination task that allows for $k = 2$ copies, sequential strategies outperform parallel strategies, i.e., $P^{PAR} < P^{SEQ}$.*

In Example 2, we have set the distribution $\{p_i\}$ to be proportional to the i^{th} digit of the constant π to emphasise that the phenomenon of sequential strategies outperforming parallel ones when the set of unitary channels forms a group does not require a particularly well-chosen non-uniform distribution. In practice, we have observed that with randomly generated distributions, optimal strategies often respect $P^{\text{PAR}} < P^{\text{SEQ}}$.

In both of the aforementioned examples, general strategies do not outperform sequential strategies. However, for the case of discrimination of unitary channels using $k = 3$ copies, we show that general strategies are indeed advantageous.

Let us start again with the case where the set of unitary channels does not form a group, but the probability distribution of the ensemble is uniform. For the following, we define $H_y \coloneqq |+_y\rangle\langle 0| + |-_y\rangle\langle 1|$, where $|\pm_y\rangle \coloneqq \frac{1}{\sqrt{2}}(|0\rangle \pm i\,|1\rangle)$, and $H_P \coloneqq |+_P\rangle\langle 0| + |-_P\rangle\langle 1|$, where $|+_P\rangle \coloneqq \frac{1}{5}(3\,|0\rangle + 4\,|1\rangle)$ and $|-_P\rangle \coloneqq \frac{1}{5}(4\,|0\rangle - 3\,|1\rangle)$.

Example 3. *For the ensemble composed by a uniform probability distribution and $N = 4$ unitary channels given by $\{U_i\} = \{\sqrt{\sigma_x}, \sqrt{\sigma_z}, \sqrt{H_y}, \sqrt{H_P}\}$, in a discrimination task that allows for $k = 3$ copies, general strategies outperform sequential strategies, and sequential strategies outperform parallel strategies. Therefore, the maximal probabilities of success satisfy the strict hierarchy $P^{PAR} < P^{SEQ} < P^{GEN}$.*

General strategies can also be advantageous for the discrimination of an ensemble composed by a non-uniform probability distribution and a set of unitary channels that forms a group. Let the set of unitary operators in Example 3 be the set of generators of a group (potentially with an infinite number of elements). Now consider the ensemble composed by such a group and a probability distribution given by $p_i = \frac{1}{4}$ for the four values of i corresponding to the four unitary operators which are the generators of the group, and $p_i = 0$ otherwise. It is straightforward to see that the maximal probabilities of successfully discriminating this ensemble would be the same as the ones in Example 3, hence satisfying $P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} < P^{\mathrm{GEN}}$. Although somewhat artificial, this example shows that advantages of general strategies are indeed possible for this kind of unitary channel ensemble.

Lastly, we now analyse whether the quantum switch, or its multiple slot generalisation, referred to it as the quantum N-switch, is useful for discriminating unitary channels. As showed in Sec. 4.3, the switchlike strategies are useful for general channel discrimination. However, in Ref. [78] we proved that quantum switchlike strategies are not useful for discriminating unitary channels (even if the unitary channels do not form a group or are not uniformly distributed). This result also holds for the multi-slot generalisation of switchlike strategies, as defined and discussed in Ref. [78].

Theorem 15 ([78])**.** *The action of the switch-like process on k copies of a unitary channel can be equivalently described by a sequential process that acts on k copies of the same unitary channel.*

Consequently, in a discrimination task involving the ensemble $\mathcal{E} = \{p_i, U_i\}_i$ composed by N unitary channels and some probability distribution, and that allows for k copies, for every switch-like tester $\{T_i^{SL}\}$, there exists a sequential tester $\{T_i^{SEQ}\}_i$ that attains the same probability of success, according to

$$\sum_{i=1}^{N} p_i \operatorname{tr}\left(T_i^{SL}\,|U_i\rangle\langle U_i|^{\otimes k}\right) = \sum_{i=1}^{N} p_i \operatorname{tr}\left(T_i^{SEQ}\,|U_i\rangle\langle U_i|^{\otimes k}\right). \tag{4.28}$$

4.4.1 An upper bound for discriminating uniformly distributed unitary channels

We now present an upper bound for the maximal probability of success for discriminating a set of d-dimensional unitary channels with general strategies when k copies are available. Our result applies to *any* ensemble of unitary channels $\mathcal{E} = \{p_i, U_i\}_{i=1}^N$ where $p_i = \frac{1}{N}$ is a uniform probability distribution.

Since general testers are the most general strategies which are consistent with a channel discrimination task, our result constitutes a true ultimate upper bound for discriminating unitary channels. Also, as we show later, there are particular choices of unitary channels for which this upper bound is attainable, showing that it cannot be improved.

Theorem 16 ([78]). *Let $\mathcal{E} = \{p_i, U_i\}_{i=1}^N$ be an ensemble composed of N d-dimensional unitary channels and a uniform probability distribution. The maximal probability of successful discrimination of a general strategy with k copies is upper bounded by*

$$P^{GEN} \leq \frac{1}{N}\,\gamma(d,k), \tag{4.29}$$

where $\gamma(d,k)$ is given by

$$\gamma(d,k) := \binom{k+d^2-1}{k} = \frac{(k+d^2-1)!}{k!(d^2-1)!}. \tag{4.30}$$

4.5 Outlook

The task of discriminating and estimating quantum channels is a fundamental question in quantum theory and has been analysed since the early stages of the field [59]. Various works have focused on scenarios involving the discrimination and estimation of uniformly distributed unitary channels that form a group [55, 59, 86]. The symmetries involved in this particular instance led to various foundational and useful results in quantum theory. However, not so much was known when analysing non-unitary channels, or even unitary channels that are not uniformly distributed, or that do not form a group.

One reason this scenario has been less explored is that the absence of symmetries often make the problem mathematically intractable. In this chapter, we overcame the mathematical complexity of this non-symmetric problem by proposing a computer-assisted method. When the dimension and number of calls is small, this computer-assisted approach enables us to identify various particularities of general channel discrimination by means of concrete examples. We believe that our results can help sharpen our intuition of the task of discriminating quantum channels beyond the very symmetric scenario.

The concrete results of the advantages of sequential and general strategies in this work focused on discrimination tasks that use $k = 2$ or 3 calls. An open question of our work is how these strategy gaps would scale with larger values of k. The preliminary results presented here indicate that the advantage of sequential over parallel strategies, and of general over sequential strategies, should be even more accentuated as a higher number of calls is allowed. This idea is supported by the intuition that the number of different ways in which one can construct sequential strategies, as compared to parallel strategies, increases with the number of slots k. Similarly, we expect such phenomenon to exist for the general case. It would then be interesting to find out exactly the rate with which these gaps grow with k.

No advantage of general strategies was found in scenarios involving discrimination of unitary channels using only $k = 2$ calls. We conjecture that, when considering $k = 2$ calls, such advantage is indeed not possible, for any number N of unitary channels. We also remark that, when considering $k = 2$ calls, Refs. [92, 93][3] prove that superchannels that preserve reversibility (i.e., transform unitary channels into unitary channels), are necessarily of the switch-like form. Intuitively, it seems plausible that the optimal general strategy for discriminating unitary channels would be one that transforms unitary channels into unitary channels. This argument of reversibility preservation, combined with our Theorem 15, could lead to a proof for our conjecture.

Furthermore, we also conjecture that, when considering $N = 2$ unitary channels, general strategies are not advantageous, for any number of calls k. In this scenario, it has been proven that sequential strategies cannot outperform parallel ones [11], and we believe this to also be the case for general strategies. The task of discriminating between two unitary channels can always be recast as the problem of discriminating a unitary operator from the identity operator. In the parallel case, the probability of successful discrimination has been shown to be related to the spread of the eigenvalues of this unitary operator [83, 94]. The proof of this result explores how sequential strategies affect the spread of the eigenvalues of unitary operators, to conclude that they cannot outperform parallel ones. A better understanding of how general strategies affect the spread of the eigenvalues of unitary operators could lead to a conclusive answer for this conjecture.

[3] Reference [92] is an original result co-written by the author of this thesis.

Chapter 5

Contributions to other areas

In this chapter, we list the articles of the author that were published after their PhD, but that were not included in the earlier chapters of this thesis.

5.1 Other contributions in higher-order quantum operations

5.1.1 Higher-order quantum operations with multiple copies of the input state

- [95] Multicopy quantum state teleportation with application to storage and retrieval of quantum programs
  F. Grosshans, M. Horodecki, M. Murao, T. Młynik, **M.T. Quintino**, M. Studziński, S. Yoshida
  arXiv:2409.10393 [quant-ph] (2024)

5.1.2 Understanding quantum and classical memory

- [96] Simple and maximally robust processes with no classical common-cause or direct-cause explanation
  M. Nery, **M.T. Quintino**, P.A. Guérin, T.O. Maciel, R. O. Vianna
  Quantum 5, 538 (2021)
  arXiv:2101.11630 [quant-ph]

- [97] Characterising the Hierarchy of Multi-time Quantum Processes with Classical Memory
  P. Taranto, **M.T. Quintino**, M. Murao, S. Milz
  Quantum 8, 1328 (2024)
  arXiv:2307.11905 [quant-ph]

- [91] Characterising memory in quantum channel discrimination via constrained separability problems
  T. Ohst, S. Zhang, C.H. Nguyen, M. Plávala, **M.T. Quintino**
  arXiv:2411.08110[quant-ph] (2024)

5.1.3 Characterising and understanding indefinite causality

- [98] Semi-device-independent certification of indefinite causal order
  J. Bavaresco, M. Araújo, Č. Brukner, **M.T. Quintino**
  Quantum 3, 176 (2019)
  arXiv:1903.10526 [quant-ph]

- [92] Consequences of preserving reversibility in quantum superchannels
  W. Yokojima, **M.T. Quintino**, A. Soeda, M. Murao
  Quantum 5, 441 (2021)
  arXiv:2003.05682 [quant-ph]

- [47] Characterising transformations between quantum objects, 'completeness' of quantum properties, and transformations without a fixed causal order
  S. Milz, **M.T. Quintino**
  Quantum 8, 1415 (2024)
  arXiv:2305.01247 [quant-ph]

- [99] Exponential separation in quantum query complexity of the quantum switch with respect to simulations with standard quantum circuits
  H. Kristjánsson, T. Odake, S. Yoshida, P. Taranto, J. Bavaresco, **M.T. Quintino**, M. Murao
  arXiv:2409.18420 [quant-ph] (2024)

- [100] Can the quantum switch be deterministically simulated?
  J. Bavaresco, S. Yoshida, T. Odake, H. Kristjánsson, P. Taranto, M. Murao, **M.T. Quintino**
  arXiv:2409.18202 [quant-ph] (2024)

5.1.4 Experiments involving indefinite causality

- [101] Experimental superposition of time directions
  T. Strömberg, P. Schiansky, **M.T. Quintino**, M. Antesberger, L. Rozema, I. Agresti, Č. Brukner, P. Walther
  Phys. Rev. Research 6, 023071 (2024)
  arXiv:2211.01283 [quant-ph]

- [102] Demonstration of a quantum SWITCH in a Sagnac configuration
  T. Strömberg, P. Schiansky, R.W. Peterson, **M.T. Quintino**, P. Walther
  Phys. Rev. Lett. 131, 060803 (2023)
  arXiv:2211.12540 [quant-ph]

- [103] Higher-order Process Matrix Tomography of a passively-stable Quantum SWITCH
  M. Antesberger, **M.T. Quintino**, P. Walther, L.A. Rozema
  PRX Quantum 5, 010325 (2024)
  arXiv:2305.19386 [quant-ph]

5.2 Bell nonlocality, measurement incompatibility, and prepare-and-measure scenarios

- [104] Most incompatible measurements for robust steering tests
  J. Bavaresco, **M.T. Quintino**, L. Guerini, T. O. Maciel, D. Cavalcanti, M.T. Cunha
  Phys. Rev. A 96, 022110 (2017)
  arXiv:1704.02994 [quant-ph]

- [105] Quantum measurement incompatibility does not imply Bell nonlocality
  F. Hirsch, **M.T. Quintino**, N. Brunner
  Phys. Rev. A 97, 012129 (2018)
  arXiv:1707.06960 [quant-ph]

- [106] Distributed sampling, quantum communication witnesses, and measurement incompatibility
  L. Guerini, **M.T. Quintino**, L. Aolita
  Phys. Rev. A 100, 042308 (2019)
  arXiv:1904.08435 [quant-ph]

- [107] Device-Independent Tests of Structures of Measurement Incompatibility
  M.T. Quintino, C. Budroni, E. Woodhead, A. Cabello, D. Cavalcanti
  Phys. Rev. Lett. 123, 180401 (2019)
  arXiv:1902.05841 [quant-ph]

- [108] Bell nonlocality with a single shot
  M. Araújo, F. Hirsch, **M.T. Quintino**
  Quantum 4, 353 (2020)
  arXiv:2005.13418 [quant-ph]

- [109] Certifying dimension of quantum systems by sequential projective measurements
  A. Sohbi, D. Markham, J. Kim, **M.T. Quintino**
  Quantum 5, 472 (2021)
  arXiv:2102.04608 [quant-ph]

- [110] Quantifying Quantumness of Channels Without Entanglement
  H.-Y. Ku, J. Kadlec, A. Černoch, **M.T. Quintino**, W. Zhou, K. Lemr, N. Lambert, A. Miranowicz, S.-L. Chen, F. Nori, Y.-N. Chen
  PRX Quantum 3, 020338 (2022)
  arXiv:2106.15784 [quant-ph]

- [111] Device-independent and semi-device-independent entanglement certification in broadcast Bell scenarios
  E.-C. Boghiu, F. Hirsch, P.-S. Lin, **M.T. Quintino**, J. Bowles
  SciPost Phys. Core 6, 028 (2023)
  arXiv:2111.06358 [quant-ph]

- [112] Classical Cost of Transmitting a Qubit
  M.J. Renner, A. Tavakoli, **M.T. Quintino**

Phys. Rev. Lett. 130, 120801 (2023)
arXiv:2207.02244 [quant-ph]

- [113] The minimal communication cost for simulating entangled qubits
  M.J. Renner, **M.T. Quintino**
  Quantum 7, 1149 (2023)
  arXiv:2207.12457 [quant-ph]

- [114] Logical possibilities for physics after MIP*=RE
  A. Cabello, **M.T. Quintino**, M. Kleinmann
  arXiv:2307.02920 [quant-ph] (2023)

- [115] Nonlocality activation in a photonic quantum network
  L. Villegas-Aguilar, E. Polino, F. Ghafari, **M.T. Quintino**, K. Laverick, I.R. Berkman, S. Rogge, L.K. Shalm, N. Tischler, E.G. Cavalcanti, S. Slussarenko, G.J. Pryde
  Nat. Commun. 15, 3112 (2024)
  arXiv:2309.06501 [quant-ph]

- [116] All incompatible measurements on qubits lead to multiparticle Bell nonlocality
  M. Plávala, O. Gühne, **M.T. Quintino**
  arXiv:2403.10564 [quant-ph] (2024)

- [117] Certifying measurement incompatibility in prepare-and-measure and Bell scenarios
  S. Egelhaaf, J. Pauwels, **M.T. Quintino**, R. Uola
  arXiv:2407.06787 [quant-ph] (2024)

5.3 Others

- [118] Implementing positive maps with multiple copies of an input state
  Q. Dong, **M.T. Quintino**, A. Soeda, M. Murao
  Phys. Rev. A 99, 052352 (2019)
  arXiv:1808.05788 [quant-ph]

Chapter 6

Discussions

This thesis aimed to contribute to the understanding of quantum information processing through the formalism of higher-order operations, focusing particularly on the transformation and discrimination of quantum channels. In our analyses, we covered strategies within the standard quantum circuit formalism, here referred to as sequential superchannels. Additionally, the mathematical framework of higher-order quantum operations allows us to analyse general superchannels, that are in agreement with quantum theory, but are not restricted to respect a definite causal order. This class of general superchannels is also analysed in this thesis. We believe that understanding the power and limitations of indefinite causality can contribute to the debate on the physical interpretation and realisation of these processes without a definite causal order. Also, as argued and justified in this thesis, due to its symmetric nature, when compared to sequential case, general superchannels are often easier to manipulate mathematically, this allows us to provide nontrivial upper bounds to problems of interest.

In the first part of this work, we examined the transformation of unitary operations, where we considered whether multiple calls of an arbitrary unitary operation could be transformed into its inverse, transpose, or complex conjugate. We have analysed parallel, sequential, and indefinite-causal-order strategies, and identified when there is a strict hierarchy between the performance of these different approaches. There, we have also shown that, in the deterministic approximation case, the problem of inverting a unitary operation with k parallel calls is equivalent to the problem of estimating a unitary operation with k calls, and equivalent to the seemingly unrelated task of $(k-1)$-port based teleportation.

The second part of this thesis centred on the problem of quantum channel discrimination. There, we studied strategies for discriminating among ensembles of channels when multiple calls are available. Traditional approaches in this area have often focused on specific cases, such as uniformly distributed unitary channels or binary discrimination. In order to go beyond these more symmetric cases, we introduce a method to rigorously obtain upper and lower bounds on the maximal probability of success that is based on semidefinite programming and computer-assisted proofs. Our approach allowed us to address a wide variety of discrimination tasks that includes non-unitary channels, unitary channels that do not have a group structure, and discriminating quantum channels that are not uniformly distributed. We hope that these results can contribute to a foundation for future work in quantum metrology, quantum sensing, communication, query complexity, and other applications where improved discrimination could be useful.

We hope that this work provides a useful perspective on higher-order operations and their possible applications in quantum information science. Indefinite causal order, as one part of this framework, may yet prove to be a valuable tool, not only for advancing our theoretical understanding but also for guiding the development of future quantum technologies.

Bibliography

[1] W. K. Wootters and W. H. Zurek, A single quantum cannot be cloned, Nature **299**, 802 (1982).

[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels, Phys. Rev. Lett. **70**, 1895–1899 (1993).

[3] C. H. Bennett and S. J. Wiesner, Communication via one- and two-particle operators on einstein-podolsky-rosen states, Phys. Rev. Lett. **69**, 2881–2884 (1992).

[4] C. H. Bennett and G. Brassard, Quantum cryptography: Public key distribution and coin tossing, Theoretical Computer Science **560**, 7 – 11 (2014), theoretical Aspects of Quantum Cryptography – celebrating 30 years of BB84.

[5] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-Independent Security of Quantum Cryptography against Collective Attacks, Phys. Rev. Lett. **98**, 230501 (2007), quant-ph/0702152.

[6] P. W. Shor, Algorithms for quantum computation: discrete logarithms and factoring, in Proceedings 35th Annual Symposium on Foundations of Computer Science (1994).

[7] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Quantum Circuit Architecture, Phys. Rev. Lett. **101**, 060401 (2008), arXiv:0712.1325 [quant-ph].

[8] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Theoretical framework for quantum networks, Phys. Rev. A **80**, 022339 (2009), arXiv:0904.4483 [quant-ph].

[9] G. Gutoski and J. Watrous, Toward a general theory of quantum games, in Proceedings of the thirty-ninth annual ACM symposium on Theory of computing (2007) pp. 565–574, quant-ph/0611234.

[10] D. Kretschmann and R. F. Werner, Quantum channels with memory, Phys. Rev. A **72**, 062323 (2005), quant-ph/0502106.

[11] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Memory Effects in Quantum Channel Discrimination, Phys. Rev. Lett. **101**, 180501 (2008), arXiv:0803.3237 [quant-ph].

[12] J. Bavaresco, M. Murao, and M. T. Quintino, Strict hierarchy between parallel, sequential, and indefinite-causal-order strategies for channel discrimination, Phys. Rev. Lett. **127**, 200504 (2021), arXiv:2011.08300 [quant-ph].

[13] M. T. Quintino, Q. Dong, A. Shimbo, A. Soeda, and M. Murao, Probabilistic exact universal quantum circuits for transforming unitary operations, Phys. Rev. A **100**, 062339 (2019), arXiv:1909.01366 [quant-ph].

[14] G. Chiribella, Optimal networks for quantum metrology: semidefinite programs and product rules, New Journal of Physics **14**, 125008 (2012), arXiv:1207.6172 [quant-ph].

[15] Q. Liu, Z. Hu, H. Yuan, and Y. Yang, Optimal strategies of quantum metrology with a strict hierarchy, Phys. Rev. Lett. **130**, 070803 (2023), arXiv:2203.09758 [quant-ph].

[16] A. Altherr and Y. Yang, Quantum metrology for non-markovian processes, Phys. Rev. Lett. **127**, 060501 (2021), arXiv:2103.02619 [quant-ph].

[17] A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, and P. Perinotti, Optimal Quantum Tomography of States, Measurements, and Transformations, Phys. Rev. Lett. **102**, 010404 (2009), arXiv:0806.1172 [quant-ph].

[18] M. Araújo, A. Feix, F. Costa, and Č. Brukner, Quantum circuits cannot control unknown operations, New Journal of Physics **16**, 093026 (2014), arXiv:1309.7976 [quant-ph].

[19] Q. Dong, S. Nakayama, A. Soeda, and M. Murao, Controlled quantum operations and combs, and their applications to universal controllization of divisible unitary operations, arXiv e-prints (2019), arXiv:1911.01645 [quant-ph].

[20] Z. Gavorová, M. Seidel, and Y. Touati, Topological obstructions to quantum computation with unitary oracles, Phys. Rev. A **109**, 032625 (2024), arXiv:2011.10031 [quant-ph].

[21] M. T. Quintino, Q. Dong, A. Shimbo, A. Soeda, and M. Murao, Reversing unknown quantum transformations: Universal quantum circuit for inverting general unitary operations, Phys. Rev. Lett. **123**, 210502 (2019), arXiv:1810.06944 [quant-ph].

[22] G. Chiribella and D. Ebler, Quantum speedup in the identification of cause-effect relations, Nature Communications **10**, 1472 (2019), arXiv:1806.06459 [quant-ph].

[23] M. Hillery, V. Bužek, and M. Ziman, Probabilistic implementation of universal quantum processors, Phys. Rev. A **65**, 022301 (2002), arXiv:quant-ph/0106088 [quant-ph].

[24] A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, and P. Perinotti, Optimal quantum learning of a unitary transformation, Phys. Rev. A **81**, 032324 (2010), arXiv:0903.0543 [quant-ph].

[25] M. Sedlák, A. Bisio, and M. Ziman, Optimal Probabilistic Storage and Retrieval of Unitary Channels, Phys. Rev. Lett. **122**, 170502 (2019), arXiv:1809.04552 [quant-ph].

[26] Y. Yang, R. Renner, and G. Chiribella, Optimal Universal Programming of Unitary Gates, Phys. Rev. Lett. **125**, 210501 (2020), arXiv:2007.10363.

[27] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Quantum computations without definite causal structure, Phys. Rev. A **88**, 022318 (2013), arXiv:0912.0195 [quant-ph].

[28] O. Oreshkov, F. Costa, and Č. Brukner, Quantum correlations with no causal order, Nature Communications **3**, 1092 (2012), arXiv:1105.4464 [quant-ph].

[29] A. Kay, Tutorial on the Quantikz Package, arXiv e-prints (2018), arXiv:1809.03842 [quant-ph].

[30] J. de Pillis, Linear transformations which preserve hermitian and positive semidefinite operators, Pacific Journal of Mathematics **23**, 129–137 (1967).

[31] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, Reports on Mathematical Physics **3**, 275–278 (1972).

[32] M.-D. Choi, Completely positive linear maps on complex matrices, Linear Algebra and its Applications **10**, 285 – 290 (1975).

[33] A. Royer, Wigner function in liouville space: A canonical formalism, Phys. Rev. A **43**, 44–56 (1991).

[34] M. Araújo, C. Branciard, F. Costa, A. Feix, C. Giarmatzi, and Č. Brukner, Witnessing causal nonseparability, New Journal of Physics **17**, 102001 (2015), arXiv:1506.03776 [quant-ph].

[35] D. Cariello, An Elementary Description of the Positive Maps with Positive Inverse, CNMAC12 .

[36] M. Wolf, Quantum Channels and Operations Guided Tour (Corolarry 6,2).

[37] M. Araújo, A. Feix, M. Navascués, and Č. Brukner, A purification postulate for quantum mechanics with indefinite causal order, Quantum **1**, 10 (2017), arXiv:1611.08535 [quant-ph].

[38] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Transforming quantum operations: Quantum supermaps, EPL (Europhysics Letters) **83**, 30004 (2008), arXiv:0804.0180 [quant-ph].

[39] T. Eggeling, D. Schlingemann, and R. F. Werner, Semicausal operations are semilocalizable, EPL (Europhysics Letters) **57**, 782–788 (2002), arXiv:quant-ph/0104027 [quant-ph].

[40] K. Życzkowski, Quartic quantum theory: an extension of the standard quantum mechanics, Journal of Physics A Mathematical General **41**, 355302 (2008), arXiv:0804.1247 [quant-ph].

[41] S. Boyd and L. Vandenberghe, Convex Optimization (Cambridge University Press, 2004).

[42] P. Skrzypczyk and D. Cavalcanti, Semidefinite Programming in Quantum Information Science, 2053-2563 (IOP Publishing, 2023).

[43] M. Ziman, Process positive-operator-valued measure: A mathematical framework for the description of process tomography experiments, Phys. Rev. A **77**, 062112 (2008), arXiv:0802.3862 [quant-ph].

[44] D. Beckman, D. Gottesman, M. A. Nielsen, and J. Preskill, Causal and localizable quantum operations, Phys. Rev. A **64**, 052309 (2001), arXiv:quant-ph/0102043 [quant-ph].

[45] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Non-markovian quantum processes: Complete framework and efficient characterization, Phys. Rev. A **97**, 012127 (2018), arXiv:1512.00589 [quant-ph].

[46] J. Wechs, H. Dourdent, A. A. Abbott, and C. Branciard, Quantum Circuits with Classical Versus Quantum Control of Causal Order, PRX Quantum **2**, 030335 (2021), arXiv:2101.08796 [quant-ph].

[47] S. Milz and M. T. Quintino, Characterising transformations between quantum objects, 'completeness' of quantum properties, and transformations without a fixed causal order, Quantum **8**, 1415 (2024), arXiv:2305.01247 [quant-ph].

[48] Q. Dong, M. T. Quintino, A. Soeda, and M. Murao, Success-or-draw: A strategy allowing repeat-until-success in quantum computation, Phys. Rev. Lett. **126**, 150504 (2021), arXiv:2011.01055 [quant-ph].

[49] M. T. Quintino and D. Ebler, Deterministic transformations between unitary operations:
 exponential advantage with adaptive quantum circuits and the power of indefinite causality, Quantum **6**, 679 (2022), arXiv:2109.08202 [quant-ph].

[50] D. Ebler, M. Horodecki, M. Marciniak, T. Młynik, M. Quintino, and M. Studziński, Optimal universal quantum circuits for unitary complex conjugation, IEEE Trans. Inf. Theory **69**, 5069–5082 (2023), arXiv:2206.00107 [quant-ph].

[51] S. Yoshida, Y. Koizumi, M. Studziński, M. T. Quintino, and M. Murao, One-to-one Correspondence between Deterministic Port-Based Teleportation and Unitary Estimation, arXiv e-prints (2024), arXiv:2408.11902 [quant-ph].

[52] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge Series on Information and the Natural Sciences (Cambridge University Press, 2000).

[53] J. Watrous, The Theory of Quantum Information (Cambridge University Press, 2018).

[54] S. D. Bartlett, T. Rudolph, R. W. Spekkens, and P. S. Turner, Quantum communication using a bounded-size quantum reference frame, New Journal of Physics **11**, 063013 (2009), arXiv:0812.5040 [quant-ph].

[55] G. Chiribella, G. M. D'Ariano, and M. F. Sacchi, Optimal estimation of group transformations using entanglement, Phys. Rev. A **72**, 042338 (2005), arXiv:quant-ph/0506267 [quant-ph].

[56] V. Buzek, M. Hillery, and R. Werner, Optimal manipulations with qubits: Universal-not gate, Phys. Rev. A **60**, R2626–R2629 (1999), arXiv:quant-ph/9901053.

[57] M. Raginsky, A fidelity measure for quantum channels, Physics Letters A **290**, 11–18 (2001), arXiv:quant-ph/0107108 [quant-ph].

[58] M. Horodecki, P. Horodecki, and R. Horodecki, General teleportation channel, singlet fraction, and quasidistillation, Phys. Rev. A **60**, 1888–1898 (1999), arXiv:quant-ph/9807091.

[59] A. S. Holevo, Probabilistic and statistical aspects of quantum theory, Vol. 1 (Springer Science & Business Media, 2011).

[60] G. Chiribella, Y. Yang, and A. C.-C. Yao, Quantum replication at the Heisenberg limit, Nature Communications **4**, 2915 (2013), arXiv:1304.2910 [quant-ph].

[61] A. Bisio, G. M. D'Ariano, P. Perinotti, and M. Sedlák, Optimal processing of reversible quantum channels, Physics Letters A **378**, 1797–1808 (2014), arXiv:1308.3254 [quant-ph].

[62] W. Harris, W. Fulton, and J. Harris, Representation Theory: A First Course, Graduate Texts in Mathematics (Springer New York, 1991).

[63] G. Chiribella and Z. Liu, Quantum operations with indefinite time direction, arXiv e-prints (2020), arXiv:2012.03859 [quant-ph].

[64] G. Chiribella and D. Ebler, Optimal quantum networks and one-shot entropies, New Journal of Physics **18**, 093053 (2016), arXiv:1606.02394 [quant-ph].

[65] J. Miyazaki, A. Soeda, and M. Murao, Complex conjugation supermap of unitary quantum maps and its universal implementation protocol, Phys. Rev. Research **1**, 013007 (2019), arXiv:1706.03481 [quant-ph].

[66] S. Ishizaka and T. Hiroshima, Asymptotic Teleportation Scheme as a Universal Programmable Quantum Processor, Phys. Rev. Lett. **101**, 240501 (2008), arXiv:0807.4568 [quant-ph].

[67] M. Mozrzymas, M. Studziński, and P. Kopszak, Optimal multi-port-based teleportation schemes, Quantum **5**, 477 (2021), 2011.09256 [quant-ph].

[68] S. Arora and B. Barak, Computational Complexity: A Modern Approach (Cambridge University Press, 2009).

[69] R. Jozsa, An introduction to measurement based quantum computation, arXiv e-prints (2005), quant-ph/0508124 [quant-ph].

[70] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Optimal Cloning of Unitary Transformation, Phys. Rev. Lett., **101**, 180504 (2008), arXiv:0804.0129 [quant-ph].

[71] M. Soleimanifar and V. Karimipour, No-go theorem for iterations of unknown quantum gates, Phys. Rev. A **93**, 012344 (2016), arXiv:1510.06888 [quant-ph].

[72] I. S. B. Sardharwalla, T. S. Cubitt, A. W. Harrow, and N. Linden, Universal Refocusing of Systematic Quantum Noise, ArXiv e-prints (2016), arXiv:1602.07963 [quant-ph].

[73] M. Navascués, Resetting Uncontrolled Quantum Systems, Phys. Rev. X **8**, 031008 (2018), arXiv:1710.02470 [quant-ph].

[74] D. Trillo, B. Dive, and M. Navascués, Translating Uncontrolled Systems in Time, Quantum **4**, 374 (2020), arXiv:1903.10568 [quant-ph].

[75] T. Odake, H. Kristjánsson, A. Soeda, and M. Murao, Higher-order quantum transformations of hamiltonian dynamics, Physical Review Research **6**, 10.1103/physrevresearch.6.l012063 (2024), arXiv:2303.09788 [quant-ph].

[76] S. Yoshida, A. Soeda, and M. Murao, Reversing Unknown Qubit-Unitary Operation, Deterministically and Exactly, Phys. Rev. Lett. **131**, 120602 (2023), arXiv:2209.02907.

[77] Y.-A. Chen, Y. Mo, Y. Liu, L. Zhang, and X. Wang, Quantum Advantage in Reversing Unknown Unitary Evolutions, arXiv e-prints (2024), arXiv:2403.04704 [quant-ph].

[78] J. Bavaresco, M. Murao, and M. T. Quintino, Unitary channel discrimination beyond group structures:
 advantages of sequential and indefinite-causal-order strategies, J. Math. Phys. **63**, 042203 (2022), arXiv:2105.13369 [quant-ph].

[79] Q. Dong, M. T. Quintino, A. Soeda, and M. Murao, The quantum switch is uniquely defined by its action on unitary operations, Quantum **7**, 1169 (2023), arXiv:2106.00034 [quant-ph].

[80] C. W. Helstrom, Quantum detection and estimation theory, Journal of Statistical Physics **1**, 231–252 (1969).

[81] A. Y. Kitaev, Quantum computations: algorithms and error correction, Russian Mathematical Surveys **52**, 1191–1249 (1997).

[82] D. Aharonov, A. Kitaev, and N. Nisan, Quantum circuits with mixed states, Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing STOC '98, 20–30 (1998).

[83] A. Acín, Statistical Distinguishability between Unitary Operations, Phys. Rev. Lett. **87**, 177901 (2001), arXiv:quant-ph/0102064 [quant-ph].

[84] A. W. Harrow, A. Hassidim, D. W. Leung, and J. Watrous, Adaptive versus nonadaptive strategies for quantum channel discrimination, Phys. Rev. A **81**, 032339 (2010), arXiv:0909.0256 [quant-ph].

[85] V. Katariya and M. M. Wilde, Evaluating the advantage of adaptive strategies for quantum channel distinguishability, Phys. Rev. A **104**, 052406 (2021), arXiv:2001.05376 [quant-ph].

[86] G. Chiribella, G. Mauro D'Ariano, and M. Roetteler, Identification of a reversible quantum gate: assessing the resources, New Journal of Physics **15**, 103019 (2013), arXiv:1306.0719 [quant-ph].

[87] M. Araújo, F. Costa, and Č. Brukner, Computational Advantage from Quantum-Controlled Ordering of Gates, Phys. Rev. Lett. **113**, 250402 (2014), arXiv:1401.8127 [quant-ph].

[88] H. Peyrl and P. A. Parrilo, Computing sum of squares decompositions with rational coefficients, Theoretical Computer Science **409**, 269–281 (2008).

[89] S. M. Rump, Verification methods: Rigorous results using floating-point arithmetic, Acta Numerica **19**, 287–449 (2010).

[90] https://github.com/mtcq.

[91] T. Ohst, S. Zhang, C. Nguyen, M. Plávala, and M. T. Quintino, Characterising memory in quantum channel discrimination via constrained separability problems, arXiv e-prints (2024), arXiv:2411.08110 [quant-ph].

[92] W. Yokojima, M. T. Quintino, A. Soeda, and M. Murao, Consequences of preserving reversibility in quantum superchannels, Quantum **5**, 441 (2021), arXiv:2003.05682 [quant-ph].

[93] J. Barrett, R. Lorenz, and O. Oreshkov, Cyclic quantum causal models, Nature Communications **12**, 885 (2021), arXiv:2002.12157 [quant-ph].

[94] G. Mauro D'Ariano, P. L. Presti, and M. G. A. Paris, Improved discrimination of unitary transformations by entangled probes, Journal of Optics B: Quantum and Semiclassical Optics **4**, S273–S276 (2002), arXiv:quant-ph/0204050 [quant-ph].

[95] F. Grosshans, M. Horodecki, M. Murao, T. Młynik, M. T. Túlio Quintino, M. Studziński, and S. Yoshida, Multicopy quantum state teleportation with application to storage and retrieval of quantum programs, arXiv e-prints (2024), arXiv:2409.10393 [quant-ph].

[96] M. Nery, M. T. Quintino, P. A. Guérin, T. O. Maciel, and R. O. Vianna, Simple and maximally robust processes with no classical common-cause or direct-cause explanation, Quantum **5**, 538 (2021), arXiv:2101.11630 [quant-ph].

[97] P. Taranto, M. T. Quintino, M. Murao, and S. Milz, Characterising the Hierarchy of Multi-time Quantum Processes with Classical Memory, Quantum **8**, 1328 (2024), arXiv:2307.11905 [quant-ph].

[98] J. Bavaresco, M. Araújo, Č. Brukner, and M. T. Quintino, Semi-device-independent certification of indefinite causal order, Quantum **3**, 176 (2019), arXiv:1903.10526 [quant-ph].

[99] H. Kristjánsson, T. Odake, S. Yoshida, P. Taranto, J. Bavaresco, M. T. Quintino, and M. Murao, Exponential separation in quantum query complexity of the quantum switch with respect to simulations with standard quantum circuits, arXiv e-prints (2024), arXiv:2409.18420 [quant-ph].

[100] J. Bavaresco, S. Yoshida, T. Odake, H. Kristjánsson, P. Taranto, M. Murao, and M. Quintino, Can the quantum switch be deterministically simulated?, arXiv e-prints (2024), arXiv:2409.18202 [quant-ph].

[101] T. Strömberg, P. Schiansky, M. T. Quintino, M. Antesberger, L. A. Rozema, I. Agresti, v. Brukner, and P. Walther, Experimental superposition of a quantum evolution with its time reverse, Phys. Rev. Research **6**, 023071 (2024), arXiv:2211.01283 [quant-ph].

[102] T. Strömberg, P. Schiansky, R. Peterson, M. T. Quintino, and P. Walther, Demonstration of a quantum SWITCH in a Sagnac configuration, Phys. Rev. Lett. **131**, 060803 (2023), arXiv:2211.12540 [quant-ph].

[103] M. Antesberger, M. T. Quintino, P. Walther, and L. A. Rozema, Higher-Order Process Matrix Tomography of a Passively-Stable Quantum Switch, PRX Quantum **5**, 010325 (2024), arXiv:2305.19386 [quant-ph].

[104] J. Bavaresco, M. T. Quintino, L. Guerini, T. O. Maciel, D. Cavalcanti, and M. T. Cunha, Most incompatible measurements for robust steering tests, Phys. Rev. A **96**, 022110 (2017), arXiv:1704.02994 [quant-ph].

[105] F. Hirsch, M. T. Quintino, and N. Brunner, Quantum measurement incompatibility does not imply Bell nonlocality, Phys. Rev. A **97**, 012129 (2018), arXiv:1707.06960 [quant-ph].

[106] L. Guerini, M. T. Quintino, and L. Aolita, Distributed sampling, quantum communication witnesses, and measurement incompatibility, Phys. Rev. A **100**, 042308 (2019), arXiv:1904.08435 [quant-ph].

[107] M. T. Quintino, C. Budroni, E. Woodhead, A. Cabello, and D. Cavalcanti, Device-independent tests of structures of measurement incompatibility, Phys. Rev. Lett. **123**, 180401 (2019), arXiv:1902.05841 [quant-ph].

[108] M. Araújo, F. Hirsch, and M. T. Quintino, Bell nonlocality with a single shot, Quantum **4**, 353 (2020), arXiv:2005.13418 [quant-ph].

[109] A. Sohbi, D. Markham, J. Kim, and M. T. Quintino, Certifying dimension of quantum systems by sequential projective measurements, Quantum **5**, 472 (2021), arXiv:2102.04608 [quant-ph].

[110] H.-Y. Ku, J. Kadlec, A. Cernoch, M. T. Quintino, W. Zhou, K. Lemr, N. Lambert, A. Miranowicz, S.-L. Chen, F. Nori, and Y.-N. Chen, Detecting quantum non-breaking channels without entanglement, PRX Quantum **3**, 020338 (2022), arXiv:2106.15784 [quant-ph].

[111] E. Boghiu, F. Hirsch, P. Lin, M. T. Quintino, and B. J., Device-independent and semi-device-independent entanglement certification in broadcast Bell scenarios, SciPost Phys. Core **6**, 028 (2023), arXiv:2111.06358 [quant-ph].

[112] M. Renner, A. Tavakoli, and M. T. Quintino, Classical cost of transmitting a qubit, Phys. Rev. Lett. **130**, 120801 (2023), arXiv:2207.02244 [quant-ph].

[113] M. Renner and M. T. Quintino, The minimal communication cost for simulating entangled qubits, Quantum **7**, 1149 (2023), arXiv:2207.12457 [quant-ph].

[114] A. Cabello, M. T. Quintino, and M. Kleinmann, Logical possibilities for physics after MIP*=RE, arXiv e-prints (2023), arXiv:2307.02920 [quant-ph].

[115] L. Villegas-Aguilar, E. Polino, F. Ghafari, M. T. Quintino, K. T. Laverick, I. R. Berkman, S. Rogge, L. K. Shalm, N. Tischler, E. G. Cavalcanti, S. Slussarenko, and G. J. Pryde, Nonlocality activation in a photonic quantum network, Nature Communications **15**, 3112 (2024), arXiv:2309.06501 [quant-ph].

[116] M. Plávala, T. Gühne, and M. T. Quintino, All incompatible measurements on qubits lead to multiparticle Bell nonlocality, arXiv e-prints (2024), 2403.10564.

[117] S. Egelhaaf, J. Pauwels, M. T. Quintino, and R. Uola, Certifying measurement incompatibility in prepare-and-measure and Bell scenarios, arXiv e-prints (2024), arXiv:2407.06787 [quant-ph].

[118] Q. Dong, M. T. Quintino, A. Soeda, and M. Murao, Implementing positive maps with multiple copies of an input state, Phys. Rev. A **99**, 052352 (2019), arXiv:1808.05788 [quant-ph].