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Certifying measurement incompatibility in prepare-and-measure and Bell scenarios

Sophie Egelhaaf[1,*], **Jef Pauwels**[1,2], **Marco Túlio Quintino**[3] **and Roope Uola**[1]

[1] Department of Applied Physics, University of Geneva, 1211 Geneva, Switzerland
[2] Constructor Institute of Technology (CIT), Geneva, Switzerland
[3] Sorbonne Université, CNRS, LIP6, F-75005 Paris, France

E-mail: sophie.egelhaaf@unige.ch

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Abstract

We consider the problem of certifying measurement incompatibility in a prepare-and-measure (PM) scenario. We present different families of sets of qubit measurements which are incompatible, but cannot lead to any quantum over classical advantage in PM scenarios. Our examples are obtained via a general theorem which proves a set of qubit dichotomic measurements can have their incompatibility certified in a PM scenario if and only if their incompatibility can be certified in a bipartite Bell scenario where the parties share a maximally entangled state. Our framework naturally suggests a hierarchy of increasingly stronger notions of incompatibility, in which more power is given to the classical simulation by increasing its dimensionality. For qubits, we give an example of measurements whose incompatibility can be certified against trit simulations, which we show is the strongest possible notion for qubits in this framework.

Keywords: quantum measurement, quantum correlations, quantum information

* Author to whom any correspondence should be addressed.

1. Introduction

Measurement incompatibility, the existence of measurements that cannot be carried out simultaneously on a single system, is a defining feature of quantum theory. It represents a radical departure from classical theory and underpins central aspects in quantum foundations, such as the resolution of the Heisenberg microscope [1], fundamental bounds in interferometry [2], and Fine's theorem [3]. More recently, measurement incompatibility has found various uses in quantum information theory [4] as a key ingredient for quantum information protocols, such as quantum key distribution [5, 6], channel verification [7], device-independent entanglement certification [8], EPR steering [9–15], Bell nonlocality [16, 17], temporal correlations [18–20], Random Access Coding [21], state discrimination tasks [22–26], distributed sampling [27], contextuality [28, 29], and programmability [30].

Incompatibility is defined at the abstract level of Hilbert spaces, as a relational property between operator measures. In practice, however, neither these abstract objects nor the relations between them, are directly observable. We only have access to the outcome statistics. While incompatibility can be formulated operationally, its direct detection hinges on the assumption that appropriately chosen sets of trusted test-states can be prepared.

Another feature of quantum theory is the ability to establish correlations between distant parties that defy classical explanations either by measuring quantum messages, as in the prepare-and-measure (PM) scenario [31–34], or by performing local measurements on pre-shared entangled states, as in the Bell scenario [16, 35]. In the Bell scenario, reaching such non-classical correlations is known to require incompatible measurements and, hence, can be used to certify incompatibility in a device-independent way, a fact studied extensively [36–39].

Here, we analyse the problem of certifying measurement incompatibility in a PM scenario. This problem has appeared in some previous works, either implicitly or explicitly [40–44]. Here, we provide a rigorous definition for the phenomenon, consistent with the previous works, and discuss some important properties. Focusing in particular on the simplest case of qubit measurements, we show that various fundamental results in the literature [40, 42, 45, 46] can be linked to our scenario, providing a better structural understanding of the concept of PM certifiable incompatibility, together with classical models and witnessing techniques for the phenomenon.

The paper is structured as follows. We present basic definitions and discuss their operational role in the second section. In the third section, we summarize the central correlation scenarios: the PM and the Bell scenario. In the fourth section, we illustrate incompatibility in the PM scenario against classical simulation protocols. More precisely, we show that all sets of qubit measurements have a four-dimensional classical model and show that the sets of qubit measurements which have a two-dimensional classical model also have a local model when used in a Bell scenario with a maximally entangled state. We further provide examples of measurement sets illustrating this. Our notion of incompatibility naturally suggests a hierarchy of increasingly stronger notions of incompatibility, corresponding to the dimensionality of the underlying classical simulation. For qubits, we show that all measurements can be simulated with quart models, and give an example of incompatible measurements that can be certified incompatible assuming trit models. We conclude the paper in the fifth section with discussion and open questions.

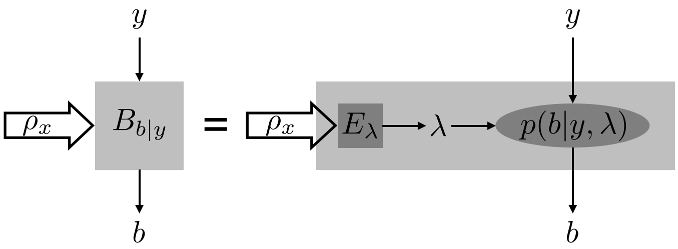

Figure 1. Operational definition of joint measurability. Joint measurability asks whether there exists a single POVM $\{E_\lambda\}_\lambda$ from which the original set of POVMs $\{B_{b|y}\}_{b,y}$ can be classically post-processed.

2. Joint measurability

A quantum measurement is modelled as a positive operator-valued measure (POVM), i.e. a collection $\{B_{b|y}\}_b \subset \mathcal{L}(\mathbb{C}_d)$ for which $B_{b|y} \geqslant 0$ and $\sum_b B_{b|y} = \mathbb{1}$. Here the index y labels the choice of measurement and b is the corresponding outcome. A central concept for sets of measurements $\{B_{b|y}\}_{b,y}$ is *joint measurability*. This asks whether there exists some further POVM from which the set can be post-processed, cf figure 1. More formally, we have the following definition:

Definition 1 (Joint measurability (JM)). A set of quantum measurements $\{B_{b|y}\}_y$ is jointly measurable (JM) if there exists a POVM $\{E_\lambda\}_\lambda$ and probability distributions $\{p(\cdot|y,\lambda)\}_{y\lambda}$ such that

$$B_{b|y} = \sum_\lambda p(b|y,\lambda)\, E_\lambda, \qquad \forall b, y \tag{1}$$

A set of measurements which is not jointly measurable is called *incompatible*. The POVM $\{E_\lambda\}_\lambda$ is called a joint or mother POVM.

This definition of joint measurability, formulated at the level of relations between operators in Hilbert space, is entirely formal. Nonetheless, in the context of quantum theory, our observations are limited to the *outcome statistics of measurements*. To make the definition more operational practical, consider a bipartite correlation experiment involving two parties, Alice and Bob. Bob possesses a device capable of implementing the various measurements $\{B_{b|y}\}_y$ based on some independently chosen input y. In each round of the experiment, Alice sends a trusted quantum state ρ_x based on some input x. Such PM experiments are characterized by a set of conditional probabilities $\{p(b|x,y)\}_{bxy}$ referred to as the correlations in the experiment. Quantum theory predicts that given these states and measurements, $p(b|x,y) = \mathrm{tr}[\rho_x B_{b|y}]$.

If the preparation device of Alice is completely trusted, we can give the practical, but equivalent definition of joint measurability:

Definition 1* (Joint measurability (JM), practical) A set of quantum measurements $\{B_{b|y}\}_{b,y}$ is jointly measurable (JM) if for every set of trusted preparations $\{\rho_x\}_x$ we can write:

$$p(b|\rho_x, y) = \sum_a p_A(a|\rho_x)\, p_B(b|y,a), \tag{2}$$

where $p_B(b|y,a)$ are classical post-processings and one requires $p_A(a|\rho_x)$ to be linear in the second argument. From Fréchet-Riesz representation theorem, it then follows that $p_A(a|\rho_x) = \text{tr}(\rho_x E_a)$ for some POVM $\{E_a\}_a$.

This equivalent definition is advantageous as it directly connects to what we can observe, namely outcomes of experiments. Specifically, it asserts that the joint measurability of the set of quantum measurements is confirmed if, for any conceivable set of trusted preparations $\{\rho_x\}_x$, the observed probabilities $p(b|\rho_x,y)$ can be expressed as a combination of the probabilities $p_A(a|\rho_x)$ related to a mother POVM $\{E_a\}_a$ and classical post-processing $p_B(b|y,a)$. However, it is important to note that this definition assumes perfect trust in Alice's device, implying that the device precisely prepares the states $\{\rho_x\}_x$.

Nonetheless, it is easy to see that the requirement of perfectly trusted preparations is not needed to check the incompatibility of various incompatible measurements. For example, consider that the preparations of Alice are noisy, i.e. instead of sending the states ρ_x, she sends the states $\eta\rho_x + (1-\eta)\frac{\mathbb{1}}{d}$. This is equivalent to assuming that the measurements performed by Bob are noisy. For any set of d-dimensional measurements $\{B_{b|y}\}_y$ and $\eta \in [0,1]$, we define the set of white noisy measurements $\{B^\eta_{b|y}\}_y$ as

$$B^\eta_{b|y} := \eta B_{b|y} + (1-\eta)\frac{\mathbb{1}}{d}\text{tr}\left(B_{b|y}\right), \ \forall b,y. \tag{3}$$

For any set of incompatible measurements $\{B^\eta_{b|y}\}_y$, the noisy set $\{B^\eta_{b|y}\}_y$ is still incompatible up to some level of noise η. This can be seen by noting that the set of jointly measurable sets of measurements is closed [47]. As a standard example, one can consider the pair of noisy spin measurements in the directions x and z. The spin measurements are given by $B_{+|1} = |+\rangle\langle+|, B_{-|1} = |-\rangle\langle-|$, $B_{+|2} = |0\rangle\langle 0|$ and $B_{-|2} = |1\rangle\langle 1|$. The resulting noisy measurements $B^\eta_{b|y}$ are jointly measurable for $\eta \leqslant 1/\sqrt{2}$ with the mother POVM $E_{i,j} = \frac{1}{4}(\mathbb{1} + \eta(i\sigma_x + j\sigma_z))$, where $i,j \in \{-1,1\}$, and otherwise incompatible [48]. The post-processing is simply marginalisation, i.e. summing over the index j gives the noisy x measurement and summing over i gives the noisy z measurement. The critical value of η at which a set of measurements $\{B^\eta_{b|y}\}_y$ becomes jointly measurable serves as a simple metric for noise robustness.

This raises the question of whether we can get a stronger certificate of joint measurability, by lifting the level of trust on Alice's preparations.

3. Device-independent certification of joint measurability

If Alice's preparations ρ_x are completely untrusted, can the parties certify the incompatibility of a set of measurements $\{B_{b|y}\}_y$? The usual approach to analysing the PM scenario (semi-)device-independently, assumes some limitation on the kind of states Alice can send, or, equivalently, the systems supported by the channel connecting the parties [49]. Indeed, if the message ρ_x can encode perfect information about the input x, the parties can reproduce any correlations p. To have a non-trivial scenario, we can limit the information that ρ_x can contain about x. A natural and common approach is to limit the dimension d of the Hilbert space of ρ_x, and to allow for shared randomness between the sender and the receiver. Note that for our purposes, this dimension is naturally equal to the dimension of the space that the measurements $\{B_{b|y}\}_y$ act on.

By definition, to certify that a set of measurements $\{B_{b|y}\}_y$ is incompatible entails proving that the correlations cannot be created by performing a single mother POVM plus classical post-processing. In a PM scenario with a single measurement setting all correlations can be simulated with classical systems of dimension n equal to the dimension d of the measurement [40]. Thus, showing incompatibility with untrusted state preparations is equivalent to demonstrating that, for *some set of states*, the resulting correlations p cannot be classically simulated. We refer to the correlations in a PM experiment p as *classical* or PM_n if p can be explained by Alice communicating n-dimensional classical systems:

Definition 2 (PM$_n$). A set of probabilities $\{p(b|x,y)\}_{b,x,y}$ can be obtained in a PM scenario with n-dimensional classical communication if there exists probability distributions p_λ, $p_A(\cdot|x,\lambda)$, $p_B(\cdot|a,y,\lambda)$, where $a \in 1,\ldots,n$ such that

$$p(b|x,y) = \sum_\lambda p_\lambda \sum_{a=1}^{n} p_A(a|x,\lambda)\, p_B(b|a,y,\lambda) \tag{4}$$

for all b,x,y.

The smallest n for which given PM correlations do have PM_n model is sometimes called the signalling dimension [50]. This leads to the following definition:

Definition 3 (PM$_n$-JM). A set of measurements $\{B_{b|y}\}_y$, $B_{b|y} \in L(\mathbb{C}_d)$ is PM_n-JM if, for every set of quantum states $\{\rho_x\}_x$, where $\rho_x \in L(\mathbb{C}_d)$, the set with probabilities given by

$$p(b|x,y) = \mathrm{tr}\left(\rho_x B_{b|y}\right) \tag{5}$$

is PM_n.

Note that in the above definitions, we can allow the dimension n of the classical simulation to be different from the dimension d of the measurements. This enables us to formulate stronger statements about the incompatibility of a set of measurements when $n > d$.

An important consequence of the above definition is that for semi-device-independent applications based on a trusted dimension $d = n$, a set of PM_n-JM measurements, even if incompatible, do not serve as a resource. This is because such applications rely on achieving a quantum–classical advantage in fixed-dimensional communication, which, by definition, cannot be generated using measurements within PM_n-JM.

In [42], the authors employ a computational polytope approximation method to demonstrate the existence of qubit measurements that are not jointly measurable (JM) but are PM_2-JM. In the next section, we will analytically prove this result by presenting several explicit examples of qubit measurements that are incompatible (not JM) but PM_2-JM.

Another natural correlation experiment which can be used to certify the incompatiblity of quantum measurements is the Bell scenario. Here, Alice and Bob, who both receive an input, respectively x and y, produce outcomes a and b, see figure 2. The parties can exchange physical systems before the experiment, but are not allowed to communicate after receiving their inputs. A Bell experiment is described by the conditional probabilities $\{p(a,b|x,y)\}_{a,b,x,y}$, referred to as correlations. The most general correlations allowed by quantum theory, are obtained by Alice and Bob performing quantum measurements, respectively $\{A_{a|x}\}_x$ and $\{B_{b|y}\}_y$ on a shared quantum state ρ_{AB}. The correlations are given by the Born rule $p(a,b|x,y) = \mathrm{tr}[\rho_{AB} A_{a|x} \otimes B_{b|y}]$.

If the correlations p can be described by Alice and Bob exchanging classical systems before the experiment, we call p *Bell local*. More precisely, we have the following definition:

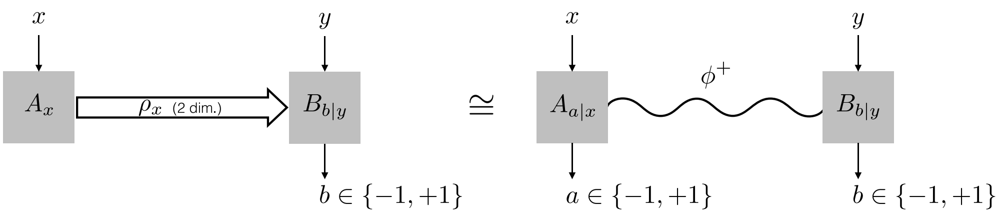

Figure 2. Left: The prepare-and-measure scenario for a qubit message and a dichotomic measurement. Right: The Bell scenario where Alice and Bob share a ϕ^+ state and perform dichotomic measurements. The set of measurements that violate the 2-dimensional classical model and the local hidden variable model, respectively, are equivalent.

Definition 4 (Bell locality). A set of probabilities $\{p(a,b|x,y)\}_{a,b,x,y}$ is Bell local if there exists probability distributions p_λ, $p_A(\cdot|x,\lambda)$, and $p(\cdot|y,\lambda)$ such that

$$p(a,b|x,y) = \sum_\lambda p_\lambda p_A(a|x,\lambda) p_B(b|y,\lambda) \tag{6}$$

for all a,b,x,y.

This naturally leads to the following definition:

Definition 5 (Bell-JM and Bell$^{\phi^+}$-JM). A set of measurements $\{B_{b|y}\}_y$, $B_{b|y} \in L(\mathbb{C}_d)$ is Bell-JM if for every quantum state $\rho_{AB} \in L(\mathbb{C}_{d'} \otimes \mathbb{C}_d)$ and every set of measurements $\{A_{a|x}\}_{a,x}$ the set with probabilities given by

$$p(a,b|x,y) = \mathrm{tr}\left(\rho_{AB} A_{a|x} \otimes B_{b|y}\right) \tag{7}$$

is Bell local.

The set of measurements $B_{b|y} \in L(\mathbb{C}_d)$ is Bell$^{\phi^+}$-JM if for every set of measurements $\{A_{a|x}\}_{a,x}$ the set with probabilities given by

$$p(a,b|x,y) = \mathrm{tr}\left(\left|\phi_d^+\right\rangle\!\left\langle\phi_d^+\right| A_{a|x} \otimes B_{b|y}\right) \tag{8}$$

is Bell local, where $\left|\phi_d^+\right\rangle := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ is the maximally entangled state.

The relationship between JM and Bell-JM has been analysed in [12, 13], which present examples of measurements which are not JM but are Bell$^{\phi^+}$-JM in a scenario where Alice's measurements are dichotomic. Later, [51] shows an example of measurements which are not JM but are Bell-JM in a scenario where Alice's measurements are dichotomic. Finally, [52] and [53] showed that there exists sets of qubit measurements which are not JM, but they are Bell-JM, regardless of what Alice measures.

4. Main results

In this section, we establish and demonstrate results pertaining to the certification of joint measurability in both PM and Bell correlations, with a specific focus on qubit scenarios. To commence, we have the following theorem [46]:

Theorem 1 (PM$_4$-JM for qubits). *Any set of qubit measurements $\{B_{b|y}\}_y$ is PM$_4$-JM.*

Proof. Reference [46] shows that all qubit correlations in a PM scenario can be simulated with two classical bits. In our language, this implies that all sets with quantum probabilities $p(b|x,y) = \mathrm{tr}(\rho_x B_{b|y})$ are PM$_4$ where $\{\rho_x\}$ is a set of qubit states. $\square$

Consequently, the following theorem represents the strongest form of incompatibility in PM qubit systems concerning classical simulability:

Theorem 2 (PM$_3$-JM for qubits). *There exists sets of qubit measurements $\{B_{b|y}\}_y$ which are not PM$_3$-JM.*

Proof. Consider a PM scenario where the eigenvectors of the Pauli matrices are prepared and 24 different qubit projective-measurements are performed. The PVMs are chosen such that their representation on the Bloch sphere are the vertices of a snub cube. We refer to appendix A for the explicit construction. The resulting correlations have been proven to not be simulable by any classical trit strategy [46]. Therefore, the set of measurements performed is not PM$_3$-JM. Hence, there exist qubit measurements $\{B_{b|y}\}_y$ which are not PM$_3$-JM. $\square$

Our subsequent theorem strongly builds upon the results and methodologies outlined in [45], which establishes a one-to-one connection between specific full-correlation dichotomic Bell scenarios and dichotomic PM scenarios.

Theorem 3 (PM$_2$-JM and Bell$^{\phi^+}$-JM for qubits). *Let $\{B^{\eta_y}_{b|y}\}_y$ be a set of qubit noisy projective measurements, that is, $B_{b|y} \in L(\mathbb{C}_2)$ and we can write $B^{\eta_y}_{0|y} = \eta_y |\phi_y\rangle\langle\phi_y| + (1-\eta_y)\frac{\mathbb{1}}{2}$, for some qubit pure state $|\phi_y\rangle$ and $\eta_y \in [0,1]$. The set of noisy measurements $\{B^{\eta_y}_{b|y}\}_y$ is Bell$^{\phi^+}$-JM if and only if it is PM$_2$-JM.*

The proof of theorem 3 is presented in appendix B. We note that the measurements used in the above theorem are exactly the ones, whose POVM elements $\{B^{\eta_y}_{b|y}\}_{b,y}$ admit a Bloch vector representation similar to quantum states, that is, we can write $B^{\eta_y}_{0|y} := \frac{1}{2}(\mathbb{1} + \vec{b}_y \cdot \vec{\sigma})$, where $\vec{b}_y \in \mathbb{R}^3$ is a vector with Euclidean norm $\left\|\vec{b}_y\right\| = \eta_y \leqslant 1$, and $\vec{b}_y \cdot \vec{\sigma} := b^1_y \sigma_x + b^2_y \sigma_y + b^3_y \sigma_z$.

We notice that the proof is constructive in the following sense: let $\{B_{b|y}\}$ be a set of measurements which are not PM$_2$, i.e. there exist a set of states $\{\rho_x\}_x$ such that $\mathrm{tr}(\rho_x B_{b|y}) \notin$ PM$_2$. Then, the behaviour $p(ab|xy) = \mathrm{tr}(\phi_2^+ A_{a|x} \otimes B_{b|y}) \notin$ Bell, where $A_{0|x} := \rho_x$ and $A_{1|x} := \mathbb{1} - \rho_x$. Conversely, if for the set of measurements $\{A_{a|x}\}_x$, we have $p(ab|xy) = \mathrm{tr}(\phi_2^+ A_{a|x} \otimes B_{b|y}) \notin$ Bell for some $\{B_{b|y}\}_y$, it holds that $\mathrm{tr}(\rho_x B_{b|y}) \notin$ PM$_2$, with the states $\rho_x = A_{0|x}$. We now discuss some consequences of theorem 3.

Corollary 1. *Let $\{X^\eta, Y^\eta, Z^\eta\}$ be the three noisy Pauli measurements. This set is JM iff $\eta \leqslant \frac{1}{\sqrt{3}}$ and PM$_2$-JM iff $\eta \leqslant \frac{1}{\sqrt{2}}$. Hence, this yields a simple example that PM$_2 -$ JM does not imply JM even for qubits.*

Proof. We start by noticing that, for the CHSH Bell inequality, if Alice and Bob share a maximally entangled state and Alice has η white noisy Pauli measurements, it is possible to obtain a CHSH violation if and only if $\eta > 1/\sqrt{2}$. To finish the proof, we simply notice that in dichotomic full correlations scenarios where one party has three measurements, the only non-trivial Bell inequality is the CHSH [54]. $\square$

Corollary 2. *Let $\left\{\Pi^{\eta}_{b|y}\right\}$ be the set of all noisy projective qubit measurements. This set is $PM_2 - JM$ iff $\eta \leqslant \frac{1}{K_3}$, where K_3 is the real Grothendieck constant of order three, which is known to respect [55]* $0.6875 \leqslant \frac{1}{K_3} < 0.6961$.

Let $\left\{\Pi^{2,\eta}_{b|y}\right\}$ be the set of all qubit noisy projective planar measurements, i.e. measurements whose Bloch vectors lie in a plane. This set is $PM_2 - JM$ iff $\eta \leqslant \frac{1}{K_2}$, where $K_2 = \sqrt{2}$ is the real Grothendieck constant of order two.

Proof. The two-qubit isotropic state $\phi^+_\eta := \eta \left|\phi^+_2\right\rangle\!\left\langle\phi^+_2\right| + (1-\eta)\frac{\mathbb{1}_4}{4}$ violates a Bell inequality with projective measurements if and only if $\eta > \frac{1}{K_3}$ [56, 57]. That is, if $\eta > \frac{1}{K_3}$, there exists measurements such that $\mathrm{tr}(\phi^+_\eta\, \Pi^A_{a|x} \otimes \Pi^B_{b|y})$ violates a Bell inequality. To finish the first part of the proof, we just notice that $\mathrm{tr}(\phi^+_\eta\, \Pi^A_{a|x} \otimes \Pi^B_{b|y}) = \mathrm{tr}(\phi^+\, \Pi^A_{a|x} \otimes \Pi^\eta_{b|y})$.

To prove the result for planar measurement, we notice that the two-qubit maximally entangled state violates a Bell inequality with projective planar measurements if and only if $\eta > \frac{1}{K_2}$ [57]. $\square$

We notice that, although stated differently, corollary 2 is equivalent to lemmas 2 and 3 from [45], and that the relationship between the Grothendieck constant with Bell and PM scenarios has been extensively discussed in [45, 55–60].

We notice that corollary 2 provides various examples of qubit measurements which are not JM but are PM_2-JM. For example, any set of η-noisy planar measurements with $\eta = 1/\sqrt{2}$ is PM_2-JM. However, if a set of $\eta = 1/\sqrt{2}$ noisy planar measurements includes two Pauli measurements and a non-Pauli one, i.e. one measurement has a Bloch vector not parallel to the other two, it is not JM. This provides several examples of incompatible measurements which cannot be certified in a PM scenario. Also, since $0.6875 \leqslant \frac{1}{K_3}$, any set of η-noisy qubit measurements with $\eta \leqslant 0.6875$ cannot be certified in PM scenario.

By using the connection between PM_2-JM and Bell$^{\phi^+}$-JM together with a known characterization of joint measurability [48], we get the following operator-inequality based connection between the three concepts.

Theorem 4. *Let $\{B^\eta_{b|y}\}_y$ be a pair of qubit white noisy projective measurements, that is $y \in \{0,1\}$. The set $\{B^\eta_{b|y}\}_y$ is JM iff it is Bell-JM, iff it is PM_2-JM, and iff $\|B^\eta_0 + B^\eta_1\| + \|B^\eta_0 - B^\eta_1\| \leqslant 2$, where $B_y := B_{0|y} - B_{1|y}$.*

Proof. The claim on JM follows from the known criterion of Busch [48] together with a suitable argument using the CHSH operator, cf appendix C, and theorem 3. $\square$

5. Discussions and open questions

A crucial question in quantum communications is under which trust assumptions we can certify quantum properties. A common and natural approach is to limit the dimension of the Hilbert space, but various other alternative assumptions have been studied [61]. This work presents such analysis for measurement resources in a PM setting, concentrating on a dimension-bounded generalisation of joint measurability. An important aspect of this generalisation is that sets of measurements simulable with certain dimensionality are effectively useless for various semi-device independent applications, such as random access codes. Here, we perform a case study for qubit measurements. These examples showcase the impact of trust assumptions on incompatibility witnesses (cf corollaries 1 and 2). This leads to a nested structure of the different sets of PM_n-JM (cf figure 3).

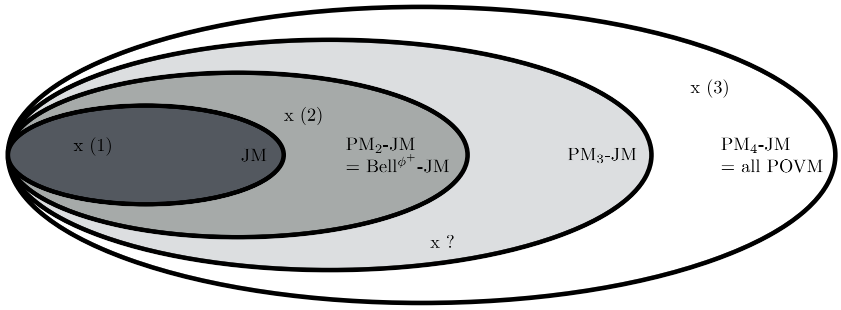

Figure 3. Schematic illustrating the relations between the different PM_n-JM sets of qubit measurements. The numbers indicate the following examples: (1) A set of POVMs having a joint measurement. (2) 3 noisy Pauli measurements with a visibility $\frac{1}{\sqrt{3}} \leqslant \eta \leqslant \frac{1}{\sqrt{2}}$ (see corollary 1). (3) Example provided in the proof of theorem 2. (?) Our results leave the existence of examples of PM_3-JM that are not PM_2-JM as an open question.

The present work suggests several interesting directions for future research. For example, a major open question is to establish a hierarchy between the set of PM_n and Bell-local measurements beyond the simple case of qubits and the ϕ^+ state. More precisely, given a set of arbitrary d-dimensional measurements, does $\{B_{b|y}\}_y \in \mathrm{PM}_n$ imply $\{B_{b|y}\}_y \in$ Bell-Local? Or, does $\{B_{b|y}\}_y \in$ Bell-Local imply $\{B_{b|y}\}_y \in \mathrm{PM}_n$? The intimate connection between the Bell and PM scenarios [62, 63] might naively suggest the existence of such a hierarchy. Notably, an invertible map exists between correlations in a Bell scenario and a subset of PM scenarios with preparations satisfying appropriate equivalences [64], see also [15]. However, the role of classical simulation dimensions in this relationship remains unclear.

In the context of qubits, the set of all possible POVMs is PM_4-JM, implying that qubit measurement incompatibility cannot be certified in a scenario where Alice can transmit two classical bits to Bob. The extension of this concept to qudits prompts intriguing questions. For instance, consider the set of all qutrit measurements $\{B_{b|y}\}_{b,y}$. Is there a classical dimension $n \in \mathbb{N}$ such that $\{B_{b|y}\}_{b,y}$ is PM_n-JM?

For qubits, does Bell^{ϕ^+}-JM imply Bell-JM in general? This question was already addressed in various previous references, e.g. [12, 13, 51–53], but it seems very hard to solve.

Another interesting direction is to generalise the definition of PM_n-JM, by asking for the existence of a simulation with dimension n only *on average* (see [65] appendix C), i.e. can the correlations be simulated by probabilistically choosing a simulation λ of dimension d_λ with probability p_λ such that $d = \sum_\lambda p_\lambda d_\lambda$? This promotes d, as a measure of joint measurability, to a continuous quantity[4]. A further interesting question is to investigate how PM_n-JM relates to signaling dimension of channels [66], classical simulation of temporal correlations [67], and how it generalizes to GPTs [34, 68].

More broadly, one could study the impact of different trust assumptions on incompatibility and other measurements resources. For example, can one formulate an analogue of energy bounds for measurements and how does a distrust assumption affect measurement simulability [69–71].

[4] While it was proven in [46] that bit simulation strategies for the set of all qubit correlations have measure zero, for trit simulations this is still an open problem.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

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Appendix A. Construction of snub cube POVM

In the following the construction of a set of projective qubit measurements is presented whose Bloch representation are the vertices of a snub cube.

The 24 vertices of a snub cube can be written as the even permutations of $\frac{1}{N}(\pm 1, \pm \frac{1}{t}, \pm t)$ with an even number of plus signs and its odd permutations with an odd number of plus signs where $t \approx 1.84$ is the tribonacci constant and $N = \sqrt{1 + \frac{1}{t^2} + t^2}$. From this the corresponding quantum qubit states can be constructed by

$$\eta_{x,y,z} = \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

where (x, y, z) are the coordinates of a chosen vertex and $i = \sqrt{-1}$. The projective measurements are constructed such that one outcome projects onto the state $\eta_{x,y,z}$ and the other outcome projects onto its complement $\mathbb{1} - \eta_{x,y,z}$.

Appendix B. Proof of theorem 3

The idea of the proof is to show an explicit bijective map between qubit PM strategies and Bell strategies with a pair of maximally entangled qubit pairs. Before presenting this construction, we establish some notation. We define the (full) correlator in a Bell scenario

$$C_{xy} := p(a = b|x, y) - p(a \neq b|x, y) \,. \tag{B1}$$

In a dichotomic Bell scenario with measurements $\{A_{a|x}\}_{a,x}$ and $\{B_{b|y}\}_{b,y}$, we define the observables

$$A_x := A_{0|x} - A_{1|x} \,, \tag{B2}$$

$$B_y := B_{0|y} - B_{1|y} \,. \tag{B3}$$

With these definitions, if the parties share the state ρ, the correlator is given by

$$C^{\rho}_{xy} = \mathrm{tr}(\rho A_x \otimes B_y) \,. \tag{B4}$$

Furthermore, for $\rho = |\phi_d^+\rangle\langle\phi_d^+|$ we have the identity

$$C_{xy}^{\phi^+} = \frac{1}{d}\mathrm{tr}\left(A_x^T B_y\right) \tag{B5}$$

where T denotes the transpose of a matrix w.r.t. the computational basis. Also, if the observable A_x and B_y respect $\mathrm{tr}(A_x) = \mathrm{tr}(B_y) = 0$, if we know the value of $C_{xy}^{\rho} = \mathrm{tr}(\rho A_x \otimes B_y)$, we can always recover the probabilities via the identity $p(a,b|x,y) = \frac{1+(-1)^{a\oplus b}C_{xy}^{\phi^+}}{4}$, where $a \oplus b$ is the addition modulo two. A proof of this identity follows from direct calculation.

Similarly, in the dichotomic PM-scenario we define the (single) correlator as

$$P_{xy} := p(b=0|x,y) - p(b=1|x,y)\,. \tag{B6}$$

If Alice sends the states ρ_x and Bob performs measurements $B_{b|y}$ the correlator is given by

$$P_{xy} := \mathrm{tr}\left(\rho_x B_y\right)\,. \tag{B7}$$

Notice that since $p(b=0|x,y) + p(b=1|x,y) = 1$, if we know the value of $P_{xy} := \mathrm{tr}(\rho_x B_y)$, we can always recover the probabilities via the identity $p(b=0|x,y) = \frac{P_{xy}+1}{2}$.

We are now ready to state lemma 1, which employs ideas from lemmas II.1, II.2, and II.3 from [60].

Lemma 1. *Let $P_{xy} := tr(\rho_x B_y)$ be qubit correlations in a dichotomic PM scenario where Alice has N_A inputs and Bob has N_B inputs with*[5] *$tr(B_y) = 0$. Let us define the new correlator P'_{xy} in a scenario where Alice includes the states given by $\mathbb{1} - \rho_x$. That is, Alice and Bob have $2N_A$ and N_B inputs respectively, and*

$$P'_{xy} := tr\left(\rho_x B_y\right) = P_{xy}, \qquad \forall x \in \{1,\ldots,N_A\}, \forall y \in \{1,\ldots,N_B\} \tag{B8}$$

$$P'_{xy} := tr\left(\left(\mathbb{1} - \rho_{(x-N_A)}\right) B_y\right) = -P_{(x-N_A)y}, \qquad \forall x \in \{N_A+1,\ldots,2N_A\}, \forall y \in \{1,\ldots,N_B\}\,. \tag{B9}$$

We now define the POVMs

$$A_{0|x} := \rho_x^T\,, \tag{B10}$$

$$A_{1|x} := \mathbb{1} - \rho_x^T\,. \tag{B11}$$

The probabilities $p(a,b|x,y) = tr(|\phi_d^+\rangle\langle\phi_d^+| A_{a|x} \otimes B_{b|y})$ are Bell local if and only if the probabilities associated to $P'_{xy} := tr(\rho_x B_y)$ are in PM_2.

Proof. We start the proof by defining the Bell correlator,

$$C'_{xy} := \mathrm{tr}\left(|\phi_2^+\rangle\langle\phi_2^+| A_x \otimes B_y\right) \qquad \forall x \in \{1,\ldots,N_A\}, \forall y \in \{1,\ldots,N_B\} \tag{B12}$$

$$C'_{xy} := -\,\mathrm{tr}\left(|\phi_2^+\rangle\langle\phi_2^+| A_{x-N_A} \otimes B_y\right) \qquad \forall x \in \{N_A+1,\ldots,2N_A\}, \forall y \in \{1,\ldots,N_B\}\,, \tag{B13}$$

[5] Assuming $\mathrm{tr}(B_y) = 0$ is equivalent to saying that Bob's POVMs are given by $B_{0|y} = \eta_y |\phi_y\rangle\langle\phi_y| + (1-\eta_y)\frac{\mathbb{1}}{2}$ for some state $|\phi_y\rangle$ and $\eta_y \in [0,1]$.

where $A_x = 2\rho_x^T - \mathbb{1}$ is the observable associated to the measurements $A_{a|x}$ defined in equation (B10).

Our next step is to show that $C'_{xy} = P'_{xy}$. This will show that, if the probabilities of P'_{xy} in the PM scenario can be obtained with qubits, we can construct a behaviour $p(a,b|x,y) = \mathrm{tr}(|\phi_d^+\rangle\langle\phi_d^+| A_{a|x} \otimes B_{b|y})$ with correlator C'_{xy}. Also, given the behaviour $p(a,b|x,y) = \mathrm{tr}(|\phi_d^+\rangle\langle\phi_d^+| A_{a|x} \otimes B_{b|y})$, we can construct a qubit PM strategy respecting $C'_{xy} = P'_{xy}$ for every x and y. For this, it is enough to notice that

$$C'_{xy} := \mathrm{tr}\left(|\phi_2^+\rangle\langle\phi_2^+| A_x \otimes B_y\right) \qquad \forall x \in \{1,\ldots,N_A\}, \forall y \in \{1,\ldots,N_B\} \tag{B14}$$

$$= \frac{1}{2}\mathrm{tr}\left(A_x^T B_y\right) \tag{B15}$$

$$= \frac{1}{2}\left(2\mathrm{tr}\left(\rho_x B_y\right) - \mathrm{tr}\left(B_y\right)\right) \tag{B16}$$

$$= \mathrm{tr}\left(\rho_x B_y\right) \tag{B17}$$

$$= P_{xy} \tag{B18}$$

$$= P'_{xy}, \tag{B19}$$

where the last equality holds because $x \in \{1,\ldots,N_A\}, \forall y \in \{1,\ldots,N_B\}$. Analogous calculation with $x \in \{N_A+1,\ldots,2N_A\}$ shows that $C'_{xy} = P'_{xy}$ for all admissible values of x and y.

We now prove that if $P'_{xy} \in \mathrm{PM}_2$, then C'_{xy} is Bell local. The intuition of the proof is that since $P'_{xy} \in \mathrm{PM}_2$, the behaviour of P'_{xy} may be obtained in a quantum scenario (potentially with shared randomness) where the measurements performed by Bob are jointly measurable. Hence, the corresponding quantum behaviour $C'_{xy} = P'_{xy}$ is necessarily Bell local. Below, we describe this proof in more details.

If $P'_{xy} \in \mathrm{PM}_2$, then for any fixed λ, by definition $p(b|x,y,\lambda) = \sum_{a=0}^{1} p_A(a|x,\lambda) p_B(b|y,a,\lambda)$. Every such strategy can be viewed as a quantum strategy where Alice sends a qubit which is diagonal in the computational basis, that is, $\rho_{x,\lambda}^c = \sum_a p_A(a|x,\lambda)\,|a\rangle\langle a|$ and Bob measures in the computational basis followed by classical post-processing, i.e. $M_{b|y,\lambda}^c := \sum_a p_B(b|y,a,\lambda)\,|a\rangle\langle a|$. Hence, $P'_{xy,\lambda} \in \mathrm{PM}_2$ can be written as

$$P'_{xy,\lambda} = \mathrm{tr}\left(\rho_{x,\lambda}^c M_{y,\lambda}^c\right), \tag{B20}$$

where the operators $\rho_{x,\lambda}^c$ and $M_{b|y,\lambda}^c$ are diagonal in the same basis. From the construction we presented in equation (B10), we see that for each λ the correlator $C'_{xy,\lambda}$ is obtained in a scenario where Alice's measurements are given by

$$A_{0|x,\lambda}^c := {\rho_{x,\lambda}^c}^T, \tag{B21}$$

$$A_{0|x,\lambda}^c := \mathbb{1} - {\rho_{x,\lambda}^c}^T. \tag{B22}$$

Since all POVM elements of Alice's measurements are diagonal in the same basis, her set of measurements is jointly measurable and cannot lead to Bell nonlocal correlations, regardless of the shared state or the measurements performed by Bob [3, 12, 13, 17], hence for each fixed λ, the correlator $C'_{xy,\lambda}$ is Bell local. Since the set of Bell local correlations is convex, convex combination of local correlators are also Bell local, that is, $\sum_\lambda \pi(\lambda) C'_{xy,\lambda}$ where $\pi(\lambda)$ is a probability distribution is necessarily Bell local.

We now prove that if the behaviour with probabilities $p(a,b|x,y) = \mathrm{tr}(|\phi_d^+\rangle\langle\phi_d^+| A_{a|x} \otimes B_{b|y})$ is Bell local, then $P'_{xy} \in \mathrm{PM}_2$. For that, we will show that, if $P'_{xy} \notin \mathrm{PM}_2$, then the correlator C'_{xy} violates a correlator Bell inequality.

If $P'_{xy} \notin \mathrm{PM}_2$, there exists a hyperplane that separates it from the classical set, that is, there exists real numbers M_{xy} such that,

$$\sum_{xy} M_{xy} P'_{xy} = Q\,, \tag{B23}$$

but, for all $P^{\mathrm{PM}_2}_{xy} \in \mathrm{PM}_2$,

$$\max_{P^{\mathrm{PM}_2}_{xy} \in \mathrm{PM}_2} \left[\sum_{xy} M_{xy} P^{\mathrm{PM}_2}_{xy}\right] = L < Q\,. \tag{B24}$$

Since, $P'_{xy} = C'_{xy}$, it follows

$$\sum_{xy} M_{xy} C'_{xy} = Q\,. \tag{B25}$$

Finally, we have to show that all local Bell correlations C^{LHV}_{xy} respect

$$\max_{C^{\mathrm{LHV}}_{xy} \in \mathrm{LHV}} \left[\sum_{xy} M_{xy} C^{\mathrm{LHV}}_{xy}\right] = L\,, \tag{B26}$$

that is, we have to show that $\sum_{xy} M_{xy} C_{xy} \leqslant L$ is a Bell inequality. This is ensured by lemma II.1 of [60], which shows that if $\max_{P^{\mathrm{PM}_2}_{xy} \in \mathrm{PM}_2} \left[\sum_{xy} M'_{xy} P^{\mathrm{PM}_2}_{xy}\right] = L$, and the coefficients' M'_{xy} can be written $M'_{xy} = M_{xy}$ for $x \in \{1, \ldots N_A\}$ and $M'_{xy} = -M_{xy}$ for $x \in \{N+A+1, \ldots 2N_A\}$, it holds that $\max_{C^{\mathrm{LHV}}_{xy} \in \mathrm{LHV}} \left[\sum_{xy} M_{xy} C^{\mathrm{LHV}}_{xy}\right] = L$. And, since our correlator P'_{xy} respects $P'_{xy} = P_{xy}$ for $x \in \{1, \ldots N_A\}$ and $P'_{xy} = -P_{xy}$ for $x \in \{N+A+1, \ldots 2N_A\}$, we can always find coefficients M'_{xy} meeting the hypothesis of lemma II.1 of [60].

□

Using lemma 1, the proof of theorem 3 follows immediately. Let $B_{b|y}$ be a set of qubit measurements which can be written as $B_{0|y} = \eta_y |\phi_y\rangle\langle\phi_y| + (1-\eta_y)\frac{\mathbb{1}}{2}$, for some qubit pure state $|\phi_y\rangle$ and $\eta_y \in [0,1]$.

If there exists qubit states ρ_x such that $p(b|x,y) = \mathrm{tr}(\rho_x B_{b|y}) \notin PM_2$, then there exist measurements $A_{a|x}$ such that $p(a,b|x,y) = \mathrm{Tr}\left(A_{a|x} \otimes B_{b|y} \phi_2^+\right)$ is not Bell local.

Appendix C. Proof of theorem 4

Definition 6. Let $A_0, A_1, B_0, B_1 \in L(\mathbb{C}_d)$ be self-adjoint operators with eigenvalues contained in $[-1,1]$. The CHSH operator is defined as

$$\mathrm{CHSH} := A_0 \otimes B_0 + A_1 \otimes B_0 + A_0 \otimes B_1 - A_1 \otimes B_1\,. \tag{C1}$$

Lemma 2. *The CHSH operator satisfies the norm inequality* $\|CHSH\| \leqslant \|B_0 + B_1\| + \|B_0 - B_1\|$.

Proof.

$$\begin{aligned}
\|CHSH\| =& \|A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1)\| && \text{(C2)}\\
\leqslant& \|A_0 \otimes (B_0 + B_1)\| + \|A_1 \otimes (B_0 - B_1)\| && \text{(C3)}\\
=& \|A_0\| \|B_0 + B_1\| + \|A_1\| \|B_0 - B_1\| && \text{(C4)}\\
\leqslant& \|B_0 + B_1\| + \|B_0 - B_1\| . && \text{(C5)}
\end{aligned}$$

□

This shows that if Bob performs measurements which satisfy $\|B_0 + B_1\| + \|B_0 - B_1\| \leqslant 2$, Alice and Bob cannot violate the CHSH inequality, regardless of the shared state, i.e. the measurements are Bell-JM. The following theorem shows that the above upper bound can be reached with $|\phi^+\rangle$.

Theorem 5. *Let $B_0, B_1 \in L(\mathbb{C}_2)$ be qubit self-adjoint operators with eigenvalues in $[-1,1]$ and $tr(B_y) = 0$. There exist self-adjoint operators $A_0, A_1 \in L(\mathbb{C}_2)$ with eigenvalues contained in $\{-1,1\}$ and a quantum state $|\psi\rangle \in \mathbb{C}_2 \otimes \mathbb{C}_2$ such that*

$$\langle \psi | \, \mathrm{CHSH} \, |\psi\rangle = \|B_0 + B_1\| + \|B_0 - B_1\| . \tag{C6}$$

Moreover, the state $|\psi\rangle \in \mathbb{C}_2 \otimes \mathbb{C}_2$ may be taken as the maximally entangled state $|\phi^+\rangle := \frac{1}{\sqrt{2}} \sum_{i=1}^{2} |ii\rangle$.

Proof. First, notice that, for any operators $A_0, A_1, B_0, B_1 \in L(\mathbb{C}_2)$, we have that

$$\left\langle \phi^+ \middle| \mathrm{CHSH} \middle| \phi^+ \right\rangle = \frac{\mathrm{tr}\left(A_0^T \otimes (B_0 + B_1)\right) + \mathrm{tr}\left(A_1^T \otimes (B_0 - B_1)\right)}{2} . \tag{C7}$$

By definition of operator norm, there exists normalised vectors $|\psi_0'\rangle, |\psi_1'\rangle \in L(\mathbb{C}_2)$ such that

$$\begin{aligned}
|\langle \psi_0' | B_0 + B_1 | \psi_0' \rangle| =& \|B_0 + B_1\| , && \text{(C8)}\\
|\langle \psi_1' | B_0 - B_1 | \psi_1' \rangle| =& \|B_0 - B_1\| . && \text{(C9)}
\end{aligned}$$

Since $\mathrm{tr}(B_y) = 0$, there exists $|\psi_0\rangle, |\psi_1\rangle \in L(\mathbb{C}_2)$ such that

$$\begin{aligned}
\langle \psi_0 | B_0 + B_1 | \psi_0 \rangle =& \|B_0 + B_1\| , && \text{(C10)}\\
\langle \psi_1 | B_0 - B_1 | \psi_1 \rangle =& \|B_0 - B_1\| . && \text{(C11)}
\end{aligned}$$

Indeed, if $\langle \psi_0' | B_0 + B_1 | \psi_0' \rangle \geqslant 0$, take $|\psi_0\rangle := |\psi_0'\rangle$. If $\langle \psi_0' | B_0 + B_1 | \psi_0' \rangle < 0$, take $|\psi_0\rangle$ as a vector which satisfies $|\psi_0\rangle\langle\psi_0| = \mathbb{1} - |\psi_0'\rangle\langle\psi_0'|$, that is, $|\psi_0\rangle$ is the universal NOT of the state $|\psi_0'\rangle$. It follows that

$$\begin{aligned}
\langle \psi_0 | B_0 + B_1 | \psi_0 \rangle &= -\mathrm{tr}\left(|\psi_0'\rangle\langle\psi_0'| (B_0 + B_1)\right)\\
&= -\langle \psi_0' | B_0 + B_1 | \psi_0' \rangle && \text{(C12)}
\end{aligned}$$

is a positive number. An analogous argument works for $|\psi_1\rangle$.

We now define $A_x =: 2|\psi_x\rangle\langle\psi_x|^T - \mathbb{1}$. We have

$$\langle\phi^+|\,\mathrm{CHSH}\,|\phi^+\rangle = \frac{\mathrm{tr}\left(A_0^T \otimes (B_0+B_1)\right) + \mathrm{tr}\left(A_1^T \otimes (B_0-B_1)\right)}{d} \tag{C13}$$

$$= \frac{2}{d}\left(\langle\psi_0|\,B_0+B_1\,|\psi_0\rangle + \langle\psi_1|\,B_0-B_1\,|\psi_1\rangle\right) - \left(\mathrm{tr}(B_0)+\mathrm{tr}(B_1)+\mathrm{tr}(B_0)-\mathrm{tr}(B_1)\right) \tag{C14}$$

$$= \frac{2}{d}\left(\langle\psi_0|\,B_0+B_1\,|\psi_0\rangle + \langle\psi_1|\,B_0-B_1\,|\psi_1\rangle\right) - 2\mathrm{tr}(B_0)\,. \tag{C15}$$

Since $\mathrm{tr}(B_0) = 0$, we have

$$\langle\phi^+|\,\mathrm{CHSH}\,|\phi^+\rangle = \frac{2}{2}\left(\langle\psi_0|\,B_0+B_1\,|\psi_0\rangle + \langle\psi_1|\,B_0-B_1\,|\psi_1\rangle\right), \tag{C16}$$

$$= \|B_0+B_1\| + \|B_0-B_1\|\,. \tag{C17}$$

□

ORCID iDs

Sophie Egelhaaf https://orcid.org/0009-0006-4769-8161
Jef Pauwels https://orcid.org/0000-0003-4440-7411
Marco Túlio Quintino https://orcid.org/0000-0003-1332-3477
Roope Uola https://orcid.org/0000-0003-3417-4458

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