All Incompatible Measurements on Qubits Lead to Multiparticle Bell Nonlocality

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Bell nonlocality is a fundamental phenomenon of quantum physics as well as an essential resource for various tasks in quantum information processing. It is known that for the observation of nonlocality the measurements on a quantum system have to be incompatible, but the question of which incompatible measurements are useful, remained open. Here we prove that any set of incompatible measurements on qubits leads to a violation of a suitable Bell inequality in a multiparticle scenario, where all parties perform the same set of measurements. Since there exists incompatible measurements on qubits which do not lead to Bell nonlocality for two particles, our results demonstrate a fundamental difference between two-particle and multiparticle nonlocality, pointing at the superactivation of measurement incompatibility as a resource. In addition, our results imply that measurement incompatibility for qubits can always be certified in a device-independent manner.

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Introduction—The study of quantum correlations in the form of entanglement [1,2], steering [3,4], and Bell nonlocality [5] has been a central topic in quantum mechanics for several decades, with Bell nonlocality being the strongest form of correlation among them. In additional to its relevance to quantum foundations [6], Bell nonlocality plays a deep role in various branches of quantum information and quantum computation [7–9]. The discovery of device-independent protocols [10] allows Bell nonlocality to be used for proving security in cryptographic protocols where the attackers may access general physical theories beyond quantum physics [11,12].

In quantum theory, Bell nonlocality requires the combination of two key ingredients, entangled states and incompatible measurements. Studies of the relationship between entanglement and Bell nonlocality date back to 1989, with Werner's seminal paper [13], which provides an explicit local hidden variable (LHV) model for quantum states, showing that entanglement is not sufficient for Bell nonlocality with projective measurements. This result was then extended to treat the most general class of measurements, described by a positive operator-valued measure (POVM) [14]. Our knowledge on how entanglement relates to nonlocality has been growing since then [15–19], leading to the phenomena of entangled states with hidden nonlocality [20,21], entangled states whose nonlocality can be superactivated by performing joint measurements on multiple copies [22,23], among others [24,25].

When compared to entanglement, the relationship between measurement incompatibility and Bell nonlocality is less understood. A first striking result came in 2009 [26], where Wolf, Perez-Garcia, and Fernandez showed that a pair of two-outcome POVMs can be used to violate a Bell inequality, if and only if it is incompatible. Later, it was shown that measurement incompatibility is necessary and sufficient for steering [27,28]. However, for general bipartite scenarios, it was proven that there are sets of measurements which are incompatible, but cannot lead to Bell nonlocality [29–31], a result which may be viewed as the existence of LHV models for measurements, in similar spirit to entangled, but Bell-local Werner states [13].

In this Letter, we address whether incompatible POVMs can violate Bell inequalities and lead to experimental demonstrations of quantum nonlocality. Contrary to the bipartite case, we show that for qubits any set of incompatible POVMs leads to Bell nonlocality and violates some multipartite Bell inequality where all parties perform the same set of measurements; see Fig. 1.

Preliminaries—We will use $\mathbb{C}_d$ to denote a complex, finite-dimensional Hilbert space, $\dim(\mathbb{C}_d) = d$. $\mathcal{L}(\mathbb{C}_d)$ will denote the set of self-adjoint (linear) operators on $\mathbb{C}_d$ and an operator $A \in \mathcal{L}(\mathbb{C}_d)$ is positive semidefinite, $A \geq 0$, when A is self-adjoint and all the eigenvalues of A are nonnegative. $\mathcal{D}(\mathbb{C}_d)$ will denote the set of density matrices which represent states of a quantum system associated to the Hilbert space $\mathbb{C}_d$, that is, $\rho \in \mathcal{L}(\mathbb{C}_d)$ if $\rho \geq 0$, and $\mathrm{Tr}(\rho) = 1$ and $\mathbb{1}$ will denote the identity operator.

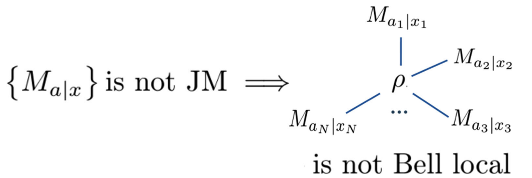

FIG. 1. In this Letter we show that, if $\{M_{a|x}\}$ is a set of qubit measurements (so-called POVMs where a and x label the inputs and outputs, respectively) which is incompatible, then there exist a number of parties $N \in \mathbb{N}$ and a quantum state $\rho \in \mathcal{D}(\mathbb{C}_2^{\otimes N})$ such that the behavior $p(a_1 \ldots a_N | x_1 \ldots x_N) = \mathrm{Tr}[\rho(M_{a_1|x_1} \otimes \ldots \otimes M_{a_N|x_N})]$ is Bell nonlocal; i.e., it violates some Bell inequality.

A POVM $\{M_a\}$ is a set of positive semidefinite operators that sum to identity, i.e., $M_a \geq 0$ and $\sum_a M_a = \mathbb{1}$. POVMs represent the most general description of a measurement that can be performed on a quantum system, and the probability of obtaining the outcome a when performing the POVM $\{M_a\}$ on the state $\rho \in \mathcal{D}(\mathbb{C}_d)$ is given by the Born's rule, which reads as $p(a) = \mathrm{Tr}(\rho M_a)$.

A set of POVMs is a set of positive semidefinite operators $M_{a|x}$ such that for every fixed value of x, $\{M_{a|x}\}$ is a POVM, in this way x may be viewed as input choice and a as the outcome. We will use $\mathsf{M} = \{M_{a|x}\}$ to denote the whole set, while $M_{a|x}$ will denote the specific operator. A set of POVMs represents several possible measurements, and one can ask whether all the measurements in the collection can be measured in a single experiment. This is captured by the definition of joint measurability [32,33].

Definition 1—A set of POVMs $\{M_{a|x}\}$ is jointly measurable (JM) or compatible if it can be decomposed as

$$M_{a|x} = \sum_\lambda p(a|x,\lambda) G_\lambda, \quad \forall\ a, x, \tag{1}$$

where $\{G_\lambda\}$ is the so-called mother POVM, obeying $G_\lambda \geq 0$ and $\sum_\lambda G_\lambda = \mathbb{1}$, and the $p(a|x,\lambda)$ are conditional probability distributions, i.e., $p(a|x,\lambda) \geq 0$, and $\sum_a p(a|x,\lambda) = 1$ for any x, λ. The set of measurements is called incompatible if it is not JM.

Quantum systems are composed by means of the tensor product. A quantum state $\rho \in \mathcal{D}(\mathbb{C}_d^{\otimes N})$ may then be viewed as a quantum state shared by N parties which we label with an integer $n \in \{1, \ldots, N\}$. We now consider a multipartite scenario where each party may perform quantum measurements on their respective local systems. These measurements may be described by the sets of POVMs $\mathsf{M}^n = \{M^n_{a_n|x_n}\}$ and the probability that the parties obtain the respective outcomes $a_1 \ldots a_N$ given the choices of measurements $x_1 \ldots x_N$ is $p(a_1 \ldots a_N | x_1 \ldots x_N) = \mathrm{Tr}[\rho(M^1_{a_1|x_1} \otimes \ldots \otimes M^n_{a_n|x_n})]$. As standard in the literature of Bell nonlocality, we refer to a set of probability distributions $\{p(a_1 \ldots a_N | x_1 \ldots x_N)\}$ as a (multipartite) behavior. For such a behavior, one can ask whether it is nonlocal or not [5].

Definition 2—A behavior $\{p(a_1 \ldots a_N | x_1 \ldots x_N)\}$ is (N-partite) Bell local if it can be written as

$$p(a_1 \ldots a_N | x_1 \ldots x_N) = \sum_\lambda p(\lambda) p_1(a_1|x_1,\lambda) \ldots p_N(a_N|x_N,\lambda), \tag{2}$$

where $\{p(\lambda)\}$ is a probability distribution and $p_n(a_n|x_n,\lambda)$ are conditional probabilities.

Bell nonlocality and joint measurability—The relationships between measurement incompatibility and Bell nonlocal correlations have been investigated over the years from foundational and practical perspectives [26–28,34–36]. It is straightforward to verify that measurement incompatibility is a necessary resource for Bell nonlocality; that is, if the set of measurements performed by $N-1$ parties are JM, the corresponding behavior $p(a_1 \ldots a_N | x_1 \ldots x_N) = \mathrm{Tr}[\rho_N(M^1_{a_1|x_1} \otimes \ldots \otimes M^n_{a_n|x_n})]$ is Bell local regardless of the quantum state ρ and regardless of the measurements performed by the other party [37]. From a more practical perspective, the necessity of measurement incompatibility for Bell nonlocality allows a device-independent certification of measurement incompatibility; that is, if a behavior $p(a_1 \ldots a_N | x_1 \ldots x_N)$ is Bell nonlocal, at least two parties are performing incompatible measurements [38,39].

The question whether measurement incompatibility is a sufficient requirement for Bell nonlocality is considerably more complex. References [26,40] shows that a pair of two-outcome POVMs $\mathsf{M} = \{M_{0|0}, M_{1|0}, M_{0|1}, M_{1|1}\}$ can be used to violate a Bell inequality, if and only if M is incompatible. However, for bipartite scenarios, this strong relationship between Bell nonlocality and measurement incompatibility turns out to be a particularity of a pair of two-outcome measurements. References [30,31] present explicit sets of incompatible qubit measurements $\{M_{a|x}\}$ such that, for any bipartite state, and any set of quantum measurements for Bob $\{N_{b|y}\}$ the behavior given by $p(ab|xy) = \mathrm{Tr}[\rho_{AB}(M_{a|x} \otimes N_{b|y})]$ is guaranteed to be Bell local (see also Ref. [29], which proves the same result under the hypothesis that Bob performs two-outcome POVMs). Hence, in a bipartite scenario, even for qubits measurement incompatibility is not a sufficient resource to observe Bell nonlocality.

Main result—We now state our main result.

Theorem 3—Let $\mathbb{C}_2$ be the qubit Hilbert space, $\dim(\mathbb{C}_2) = 2$, and let $\mathsf{M} = \{M_{a|x}\}$ be a finite set of qubit POVMs; that is $M_{a|x} \in \mathcal{L}(\mathbb{C}_2)$, $M_{a|x} \geq 0$ and $\sum_a M_{a|x} = \mathbb{1}$; $a \in \{1, \ldots, n_a\}$, $x \in \{1, \ldots, n_x\}$ and $n_a, n_x \in \mathbb{N}$. If M is incompatible, then there exists an $N \in \mathbb{N}$ and a state $\rho \in \mathcal{D}(\mathbb{C}_2^{\otimes N})$ such that the behavior $p(a_1 \ldots a_N | x_1 \ldots x_N) = \mathrm{Tr}[\rho(M_{a_1|x_1} \otimes \ldots \otimes M_{a_N|x_N})]$ is multipartite Bell nonlocal.

In short, this theorem states that, for qubits, if a set of POVMs M is incompatible, there exists a quantum state ρ such that if the N parties perform exactly the same measurements M on ρ, the resulting behavior leads to a violation of some Bell inequality. The detailed proof of Theorem 3 is delegated to Supplemental Material [41]; it relies on the following four key steps. (a) First, we reinterpret the notion of Bell nonlocality of a behavior. In fact, Bell locality in the sense of Def. 2 means that the behavior can be seen as a separable state in the tensor product space of conditional probabilities. This is not the standard notion of separability of quantum states [2]; rather it is a notion of separability for generalized probabilistic theories [42,43]. (b) Second, we consider the set of POVMs $\mathsf{M} = \{M_{a|x}\}$ containing n_x measurements. Since $p(a|x) = \mathrm{Tr}(\rho M_{a|x})$ is a conditional probability, we can consider M as a map from density matrices to conditional probabilities. Ideas similar to the Choi-Jamiołkowski isomorphism [50–52] associate to this map a state of the tensor product of positive operators and conditional probabilities. We show that if M as a map does not correspond to a separable state in this picture, then there must exist a certain positive, but not completely positive map Φ on the space of quantum states, arising from a witness-like construction. (c) Third, we consider general maps on qubits. We prove that a positive map that is entanglement annihilating must be completely positive, which is an extension of a result in Ref. [44]. This connects to (a) and (b) as follows: if a set of POVMs M applied to N qubits gives always Bell-local, i.e., separable according to (a), behaviors, then maps like Φ must be entanglement annihilating. It follows that in this case the map M in the sense of (b) must be separable; otherwise there is a contradiction to Φ not being completely positive. (d) In the final step we close the argument. We show that if M is a separable map in (b), then the set of POVMs is JM. This follows by interpreting the separable decomposition in (b) as a mother POVM for all the POVMs in M, similar arguments are known [53–55]. This proves the Theorem: if the POVMs in the set M are incompatible, then there must be an N where the behavior generated by these measurements on qubits becomes Bell nonlocal. If this were not the case, then according to (c), M is a separable map in (b) and we would arrive at a contradiction.

An immediate question to ask is whether for a given set of POVMs M one can identify necessary or sufficient number of parties N for a violation of some Bell inequality. The most feasible way to obtain analytic bounds would be to investigate whether the positive but not completely positive map constructed in step (b) is locally entanglement annihilating for N parties: if the map Φ is not locally entanglement annihilating, then also M is not locally entanglement annihilating and thus some Bell inequality must be violated. For $N = 2$ such conditions are known for specific classes of positive maps [56,57], so they are not directly applicable to our case. In general deciding whether a given map is locally entanglement annihilating is related to whether a related map is tensor stable positive [45], which was recently conjectured to be undecidable problem [58]. We will later show that this problem may be approached numerically for certain scenarios.

It is natural to ask whether our result can be extended to $d > 2$ dimensions. In the following we prove that, if our result does not hold for an arbitrary dimension d, then there exist bipartite qudit states $\rho_{AB} \in \mathcal{D}(\mathbb{C}_{d^2})$ which has a nonpositive partial transpose (NPT); i.e., $\rho_{AB}^{T_B}$ is not positive, and ρ_{AB} is bound entangled; in other words, a maximally entangled state cannot be distilled from many copies of ρ_{AB} [59]. The existence of NPT bound entangled states is a long-standing important open problem in quantum information, which has been studied from various researchers and different perspectives [45,46,60–64]. Hence, we expect that generalizing our main theorem for $d > 2$ to be a very hard mathematical problem.

Theorem 4—If there exist a set of d-dimensional incompatible POVMs $\mathsf{M} = \{M_{a|x}\}$ such that for all $N \in \mathbb{N}$ and all states $\rho \in \mathcal{D}(\mathbb{C}_d^{\otimes N})$, the behavior $p(a_1 \ldots a_N | x_1 \ldots x_N) = \mathrm{Tr}[\rho(M_{a_1|x_1} \otimes \ldots \otimes M_{a_N|x_N})]$ is Bell local, then there exists an NPT bound entangled bipartite quantum state in dimension d^2.

If the conditions of the Theorem 4 hold, then one can use the ideas from the proof of Theorem 3 to show that there must be maps which are entanglement annihilating, but not entanglement breaking, which implies the existence of NPT bound entanglement according to Ref. [46]; details can be found in Supplemental Material [41].

Superactivation of measurement incompatibility for Bell nonlocality—As previously mentioned, Refs. [30,31] present sets of qubit dichotomic measurements $\mathsf{M} = \{M_{a|x}\}$ which are not jointly measurable, but, for any quantum state $\rho_{AB} \in \mathcal{D}(\mathbb{C}_2 \otimes \mathbb{C}_d)$ and any sets of measurements $\{B_{b|y}\}$, the bipartite behavior $p(ab|xy) = \mathrm{Tr}[\rho_{AB}(M_{a|x} \otimes B_{b|y})]$ is Bell local. This result is interpreted as existence of incompatible measurements which are useless for Bell nonlocality in bipartite scenarios. When multipartite scenarios are considered, our main result shows that all sets of incompatible qubit measurements can lead to Bell nonlocality. One might then argue that the sets of incompatible but Bell local measurements $\mathsf{M} = \{M_{a|x}\}$ presented in Refs. [30,31] have some nonlocality that is activated in a scenario with multiple parties.

Interestingly, Ref. [30] shows that, if $\mathsf{M} = \{M_{a|x}\}$ is a set of measurements leading to Bell local correlations on any bipartite scenario, this set of measurements cannot lead to genuine multipartite Bell nonlocality if more parties are considered. Let us, for concreteness, present the argument for the tripartite case. Let $p(abc|xyz)$ be the probabilities of a tripartite behavior. For the given measurements, it must be by assumption Bell local in the A/BC bipartition. That is, it can be written as

$p(abc|xyz) = \sum_\lambda p(\lambda)p_A(a|x,\lambda)p_{BC}(bc|yz,\lambda)$. There exists various non-equivalent notions of genuine multipartite Bell nonlocality [65], but the loosest one is given by Svetlichny [66], and states that a tripartite behavior is not genuine tripartite Bell local if it can be written as a convex combination of behaviors which are Bell local in a bipartition. Hence, if a behavior is Bell local in the A/BC bipartition, it cannot be genuine multipartite nonlocal.

In combination with out results this has interesting consequences. Let $\mathsf{M} = \{M_{a|x}\}$ be the set of qubit measurements presented in Refs. [30,31]. For any $N \in \mathbb{N}$ and any quantum state $\rho \in \mathcal{D}(\mathbb{C}_2^{\otimes N})$, the behavior $p(a_1 \ldots a_N|x_1 \ldots x_N) = \mathrm{Tr}[\rho(M_{a_1|x_1} \otimes \ldots \otimes M_{a_n|x_n})]$ is Bell local in every bipartition of the form $1/N-1$, that is, where one side of the partition has a single party. However, it follows from our main result that there exists some $N \in \mathbb{N}$, and a quantum state $\rho \in \mathcal{D}(\mathbb{C}_2^{\otimes N})$ such that $p(a_1 \ldots a_N|x_1 \ldots x_N) = \mathrm{Tr}[\rho(M_{a_1|x_1} \otimes \ldots \otimes M_{a_n|x_n})]$ is Bell nonlocal, but Bell local in all 1 vs $(N-1)$ partitions, hence not genuine multipartite nonlocal. A similar phenomenon has been phrased in previous literature as *anonymous nonlocality* [67] and finds applications in multipartite secret sharing. Note that this phenomenenon also occurs in entanglement theory, where quantum states can be separable for any bipartition, but not fully separable [62,68,69].

Quantifying Bell incompatibility—One may quantify measurement incompatibility by how robust it is against white noise [70–73] via the η depolarizing quantum channel, defined as $D_\eta(\rho) \coloneqq \eta\rho + (1-\eta)\mathrm{Tr}(\rho)\mathbb{1}$. If $\mathsf{M} = \{M_{a|x}\}$ is a set of POVMs, its noisy version is the POVM defined as $\mathsf{M}^{(\eta)} \coloneqq \{D_\eta(M_{a|x})\}$. The incompatibility robustness of a POVM M is the maximum visibility parameter η such that $\mathsf{M}^{(\eta)}$ is JM. More precisely, the maximum value of $\eta \in [0,1]$ such that $\mathsf{M}^{(\eta)}$ is jointly measurable; i.e., $\eta^{\mathrm{JM}}(\mathsf{M}) \coloneqq \max \eta$ such that $\mathsf{M}^{(\eta)}$ is JM. Since we set $\eta \in [0,1]$, a set of POVMs M is jointly measurable if and only if $\eta^{\mathrm{JM}}(\mathsf{M}) = 1$.

Motivated by our main result, we introduce the definition of k-partite Bell joint measurability, which quantifies the critical visibility $\eta \in [0,1]$ for which a set of POVMs $\mathsf{M}^{(\eta)}$ becomes Bell local in any k-partite scenario regardless of the shared state, where all parties perform the (same) measurements $\mathsf{M}^{(\eta)}$.

Definition 5—Let $\mathsf{M} = \{M_{a|x}\}$ be a set of POVMs acting in $\mathbb{C}_d$. Its noisy k-partite Bell joint measurability $\eta_k^{\mathrm{Bell}}(\mathsf{M}) \in [0,1]$ is defined as $\eta_k^{\mathrm{Bell}}(\mathsf{M}) \coloneqq \max \eta$ such that $\forall\ \rho \in \mathcal{D}(\mathbb{C}_d^{\otimes k})$, $\mathrm{Tr}[\rho(M^{(\eta)}_{a_1|x_1} \otimes \ldots \otimes M^{(\eta)}_{a_k|x_k})]$ is Bell local.

Since only incompatible measurements can lead to Bell nonlocality, it follows that if M is JM, we have $\eta_k^{\mathrm{Bell}}(\mathsf{M}) = 1$. Generally, $\eta_{\mathrm{JM}}(\mathsf{M}) \leq \eta_k^{\mathrm{Bell}}(\mathsf{M})$ for any $k \in \mathbb{N}$. Also, for any set of POVMs M and any $k \in \mathbb{N}$ it holds that $\eta_{k+1}^{\mathrm{Bell}}(\mathsf{M}) \leq \eta_k^{\mathrm{Bell}}(\mathsf{M})$, this is true because if the marginal behavior $\sum_{a_i} \mathrm{Tr}[\rho(M_{a_1|x_1} \otimes \ldots \otimes M_{a_k|x_k})]$ is Bell nonlocal, the full behavior $\mathrm{Tr}[\rho(M_{a_1|x_1} \otimes \ldots \otimes M_{a_{k+1}|x_{k+1}})]$ is also nonlocal. By making use of Definition 5, our main result may be reformulated as follows.

Corollary 6—Let $\mathbb{C}_2$ be the qubit Hilbert space and let $\mathsf{M} = \{M_{a|x}\}$ be a finite set of incompatible POVMs. Then $\lim_{k\to\infty} \eta_k^{\mathrm{Bell}}(\mathsf{M}) = \eta^{\mathrm{JM}}(\mathsf{M})$.

We now present some examples to illustrate the definition of noisy k-partite Bell joint measurability. Let us start with the set of POVM $\mathsf{P} = P_{a|x}$ corresponding to the Pauli measurements Z and X. More precisely, we define $P_{0|0} \coloneqq |0\rangle\langle 0|, P_{1|0} \coloneqq |1\rangle\langle 1|, P_{0|1} \coloneqq |+\rangle\langle +|, P_{1|1} \coloneqq |-\rangle\langle -|$, where $|0\rangle, |1\rangle, |+\rangle, |-\rangle$ are the eigenvectors of the Pauli Z and X matrices, respectively. The white noisy robustness for a pair of Pauli measurements is $\eta_{JM}(\mathsf{P}) = 1/\sqrt{2}$ [32], we prove that $\eta_2^{\mathrm{Bell}}(\mathsf{P}) = (1/\sqrt[4]{2}) \approx 0.8409$ in the End Matter. Note that the two values do not coincide, since here Alice and Bob have to perform the same set of measurements, contrary to Ref. [26].

Using similar ideas, we can also show that $\eta_3^{\mathrm{Bell}}(\mathsf{P}) \approx 0.7938$, proving that for a pair of Pauli measurements, $\eta_2^{\mathrm{Bell}}(\mathsf{P}) > \eta_3^{\mathrm{Bell}}(\mathsf{P})$. To attain this goal, we first notice that we know all the tight Bell inequalities in tripartite scenario where all parties perform two dichotomic measurements [74–77]. Then, we fix our measurements as η white noisy pair of Pauli measurements and evaluate the eigenvalues of all possible Bell operators.

We now focus our attention to the set of so-called trine measurements presented in Ref. [31], which can be defined as $T_{a|x} \coloneqq \frac{1}{2}(\mathbb{1} + (-1)^a \vec{v}_x \cdot \vec{\sigma})$ where $\vec{v}_x = [\cos(2\pi x/3), 0, \sin(2\pi x/v)]$, and $\vec{v}_x \cdot \vec{\sigma} \coloneqq \cos(2\pi x/3)X + \sin(2\pi x/3)Z$, with X, Z being Pauli matrices. Reference [31] shows $\eta^{\mathrm{JM}}(\mathsf{T}) = \frac{2}{3}$ and for $\eta^* = 0.67 > \frac{2}{3}$ the set $\mathsf{T}^{\eta^*} = D_\eta(T_{a|x})$ is incompatible, but $\mathrm{Tr}[\rho_2(T^{\eta^*}_{a|x} \otimes B_{b|y})]$ is Bell local for all $\rho_2 \in \mathcal{D}(\mathbb{C}_2 \otimes \mathbb{C}_2)$ and all sets of POVMs $\{B_{b|y}\}$. Following our main result, T^{η^*} is a set of measurements whose nonlocality can be superactivated by considering multipartite scenarios.

Up to relabelling, the bipartite Bell scenario with three inputs, and two outputs has only two tight Bell inequalities, CHSH inequality and the I3322 inequality [78,79]. In order to use the CHSH inequality, we only make use two out of the three measurements of the trine given by $\{T^\eta_{a|x}\}$, and direct calculation shows that the maximal eigenvalue of its associated CHSH operator is $\eta^2\sqrt{7}$. Hence, we see that $\eta^2\sqrt{7} > 2$ iff $\eta > (\sqrt{2}/\sqrt[4]{7}) \approx 0.8694$. We can show that the I3322 inequality and its relabelling are not violated for any state ρ by setting $\eta = (\sqrt{2}/\sqrt[4]{7})$ and evaluating the eigenvalues of its associated Bell operator. This shows that $\eta_2^{\mathrm{Bell}}(\mathsf{T}) = (\sqrt{2}/\sqrt[4]{7})$.

By making use of a computational method detailed in the End Matter, we could also show that when three parties are considerd, we have that $\eta_3^{\mathrm{Bell}}(\mathsf{T}) < 0.8007$, hence $\eta_3^{\mathrm{Bell}}(\mathsf{T}) < \eta_2^{\mathrm{Bell}}(\mathsf{T})$. Finally, we remark that using the methods for symmetric multipartite Bell inequalities presented in Ref. [80], one may certify nonlocality in scenarios with a large number of parties N, hence to obtain nontrivial (numerical and analytical) lower bounds for $\eta_N^{\mathrm{Bell}}(\mathsf{M})$ for large N.

Implications for multipartite Bell nonlocality—It is known that the set of all qubit POVMs is jointly measurable if and only if their white noisy visibility is smaller than or equals one have, that is, if M be the set of all qubit POVMs, $\eta^{\mathrm{JM}}(\mathsf{M}) = 1/2$ [81,82]. Hence, by our main theorem, it follows that, if $\eta > 1/2$, there exists an $N-$ partite qubit state ρ and a set of measurements $\{M_{a|x}\}$ such that $\mathrm{Tr}[\rho D_\eta(M_{a_1|x_1}) \otimes \ldots D_\eta(M_{a_N|x_N})] = \mathrm{Tr}[D_\eta^{\otimes N}(\rho) M_{a_1|x_1} \otimes \ldots M_{a_N|x_N}]$ is Bell nonlocal.

Corollary 7—If $\eta > \frac{1}{2}$, there exists a $N \in \mathbb{N}$, and a N qubit state $\rho \in \mathcal{D}(\mathbb{C}_2^{\otimes N})$ such that the state $D_\eta^{\otimes N}(\rho)$ is entangled and violates a Bell inequality.

This demonstrates that there must be quantum states whose Bell nonlocality is very robust against noise. Note that GHZ states, if the standard Mermin inequality [83] is used, lead to a robustness corresponding to $\eta > (1/\sqrt{2})$ only. So, this predicted effect is stronger than the exponential violation of local realism from the Mermin inequality.

Discussion and open questions—We have proven a very strong link between qubit measurement compatibility and multipartite Bell scenarios. More precisely, we have shown that all sets of incompatible measurements lead to Bell nonlocality if sufficiently many parties are considered. For this, it is enough that all parties perform exactly the same measurement. When contrasted with the bipartite case [30,31], our results show that the nonlocality of incompatible measurements may be superactivated by considering multipartite scenarios. Our main result induced a novel quantifier for the Bell nonlocality and the incompatibility of POVMs, based on how many parties are required to obtain Bell nonlocality, a concept which was also explored in this work.

An immediate open question is to identify the multiparticle quantum states ρ that are used to prove Theorem 3. Can they be restricted to specific known families of states, such as GHZ states or Dicke states? Similarly, Corollary 7 proves the existence of multiqubit states with an extreme robust violation of local realism. Can these states be identified and used as a resource for information processing tasks? Or can we at least bound the number of parties N?

A more challenging open question would be to generalize our results beyond qubits without the assumption of nonexistence of NPT bound entangled states, but this may turn out to be an extremely tough problem to solve. But more importantly, it was recently observed that even an arbitrary small amount of Bell nonlocality is a resource for device-independent quantum key distribution [84]. One can thus ask whether any incompatible measurements on qubits (or even qudits) can be used for some form of multipartite device-independent quantum key distribution. If this turns out to be the case, we are likely to obtain device-independent quantum key distribution protocol for many parties with high robustness to local noise.

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End Matter

Pauli X and Z measurements—In a bipartite scenario where each part has two dichotomic measurements, all tight Bell inequalities are equivalent to the CHSH inequality [37], hence we just need to check when noisy Pauli measurements may lead to a Bell violation of this inequality. We then consider that both Alice and Bob perform the set of two Pauli measurements, P and that $P_x \coloneqq P_{0|x} - P_{1|x}$ is the observable associated to the POVM $\{P_{0|x}, P_{1|x}\}$. The behavior $p(a_1 a_2|x_1 x_2) = \mathrm{Tr}[\rho(P_{a_1|x_1} \otimes P_{a_2|x_2})]$ is Bell local for all states $\rho \in \mathcal{D}(\mathbb{C}_2 \otimes \mathbb{C}_2)$ if and only if the operator $\mathrm{CHSH} \coloneqq P_0 \otimes P_0 + P_0 \otimes P_1 + P_1 \otimes P_0 - P_1 \otimes P_1$ has an eigenvalue strictly greater than 2 [85]. Now notice that for noisy Pauli measurements $D_\eta(P_{a|x})$, the associated observable reads as ηP_x hence, the CHSH operator takes the form of $\eta^2\mathrm{CHSH}$, which has its maximal eigenvalue given by $\eta^2 2\sqrt{2}$ [86–88], a quantity which is greater than 2 if $\eta^2 > (1/\sqrt{2})$, hence $\eta_2^{\mathrm{Bell}}(\mathsf{P}) = (1/\sqrt[4]{2}) \approx 0.8409$.

Trine measurements—The full list of Bell inequalities for scenarios with three parties, three inputs, and two outputs is not known and finding these Bell inequalities is computationally challenging [75]. Since the Bell inequalities are unknown, finding the optimal state ρ is not an obvious task. We can however obtain an upper bound on $\eta_3^{\mathrm{Bell}}(\mathsf{T})$, for that it is enough to find a tripartite quantum state ρ such that $\mathrm{Tr}[\rho(T^\eta_{a|x} \otimes T^\eta_{b|y} \otimes T^\eta_{c|z})]$, and verifying if a behavior is Bell local can be done by linear programming [5]. By choosing the tripartite state $\rho_3 \coloneqq |\mathrm{GHZ}_Y\rangle\langle\mathrm{GHZ}_Y|$, where $|\mathrm{GHZ}_Y\rangle \coloneqq [(|+_y\rangle|+_y\rangle|+_y\rangle + |-_y\rangle|-_y\rangle|-_y\rangle)/\sqrt{2}]$ and $|\pm_y\rangle \coloneqq [(|0\rangle \pm i|1\rangle)/\sqrt{2}]$ are the eigenvectors of the Pauli Y operator, linear programming shows that for $\eta = 0.8007$, the behavior $\mathrm{Tr}[\rho_3(T^\eta_{a|x} \otimes T^\eta_{b|y} \otimes T^\eta_{c|z})]$ is Bell NL, hence $\eta_3^{\mathrm{Bell}}(\mathsf{T}) < 0.8007$, and we have that $\eta_3^{\mathrm{Bell}}(\mathsf{T}) < \eta_2^{\mathrm{Bell}}(\mathsf{T})$. (All computational code used here is available at [89].) Additionally, in the case of three parties and the set of POVMs $\mathsf{T}^{\eta *}$, one can prove that for any state $\rho_3 \in \mathcal{D}(\mathbb{C}_2^{\otimes 3})$ we have $\mathrm{Tr}[\rho_3(T^{\eta^*}_{a|x} \otimes T^{\eta^*}_{b|y} \otimes T^{\eta^*}_{c|z})] = p^3 \mathrm{Tr}[\rho_3(T_{a|x} \otimes T_{b|y} \otimes T_{c|z})] + (1-p^3) p_L(abc|xyz)$, where $p_L(abc|xyz)$ is a behavior with local hidden variable model and $p \approx 10^{-2}$; see Supplemental Material [41] for a proof. Hence, it is likely that $\eta_3^{\mathrm{Bell}}(\mathsf{T}^{\eta^*}) = 1$, and we need more parties to superactivate the nonlocality of T^{η^*}.