Success-or-Draw: A Strategy Allowing Repeat-Until-Success in Quantum Computation

Qingxiuxiong Dong,[1,*] Marco Túlio Quintino,[1,2,3,†] Akihito Soeda,[1,‡] and Mio Murao[1,4,§]

[1] *Department of Physics, Graduate School of Science, The University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan*

[2] *Vienna Center for Quantum Science and Technology (VCQ), Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria*

[3] *Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria*

[4] *Trans-scale Quantum Science Institute, The University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan*

(Received 7 November 2020; accepted 15 March 2021; published 16 April 2021)

The repeat-until-success strategy is a standard method to obtain success with a probability that grows exponentially with the number of iterations. However, since quantum systems are disturbed after a quantum measurement, how to perform repeat-until-success strategies in certain quantum algorithms is not straightforward. In this Letter, we propose a new structure for probabilistic higher-order transformation named success-or-draw, which allows a repeat-until-success implementation. For that we provide a universal construction of success-or-draw structure that works for any probabilistic higher-order transformation on unitary operations. We then present a semidefinite programming approach to obtain optimal success-or-draw protocols and analyze in detail the problem of inverting a general unitary operation.

DOI: 10.1103/PhysRevLett.126.150504

Introduction.—Quantum algorithms are an inevitable element for exploiting the potential of quantum computation [1–3]. In many quantum algorithms, a unitary operation characterizing the problem and its related quantum operations, such as its inverse operation, are used as subroutines. Quantum supermaps describe such relationships between quantum operations and are used for analyzing higher-order transformations between quantum operations [4–6]. In spite of their concrete formalism, many "useful" supermaps, such as cloning unitary operations [7], inverting unitary operations [8–12], controlling unitary operations [13], and unitary learning [11,14], are not physically implementable in an exact and deterministic manner. To perform such supermaps, two types of relaxation are usually considered: the approximate transformation and the probabilistic transformation. In addition to these relaxations, adding certain resources is also considered, especially by allowing multiple calls of an input quantum operation. In the quantum circuit implementation, multiple calls are achievable by using the corresponding quantum circuit multiple times. With the assumption of multiple calls, the strategy to approximate supermaps has an advantage because it is always possible to perform process tomography [15] to obtain a classical description of the input quantum operation, calculate the output quantum operation of the supermap, and implement the output quantum operation according to the classical description. On the other hand, it is not known in general whether we can perform a supermap probabilistically but exactly, even if an arbitrary but finite number of calls are allowed.

Quantum process tomography [15] allows a universal and approximate implementation of quantum supermaps, but the figure of merit, usually the average fidelity F, is expected to scale as $1-1/\mathrm{poly}(N)$ given N calls of the input operation. The probabilistic strategy, on the other hand, can achieve a success probability that converges to 1 exponentially, if it is possible to perform independent trials such as in a repeat-until-success protocol. That is, if we can perform a probabilistic supermap with probability p using "a unit of resources" such as one quantum operation, we can perform this probabilistic supermap with probability $1-(1-p)^N$ using N units of resources. However, the resources required to perform a supermap are not only the input quantum operations; the input state that the output quantum operation of the supermap is applied on should also be counted as a resource. In quantum mechanics, transformations usually disturb quantum states [16,17], and the input state of a probabilistic supermap is usually changed regardless of success or failure; that is, the input state is lost after a trial of a supermap. Also, the cloning of a quantum state is forbidden by the no-cloning theorem [18]. Thus, it is not possible to simply perform independent trials. On the other hand, while allowing multiple copies of an input state may help in certain tasks [19], it is difficult to realize in many cases. For this reason, we consider probabilistic supermaps under the following assumption, which is also a well-studied scenario in many previous researches [7–11,13,14,20–22]: multiple calls of an input quantum operation are allowed, and only a single use of an input state is available.

In this Letter, we propose a structure of a probabilistic supermap called a "success-or-draw" structure as Fig. 1(a)

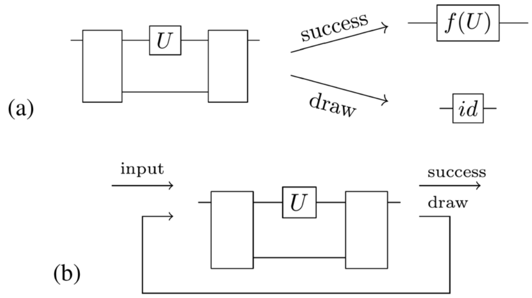

FIG. 1. Success-or-draw supermaps are ones in which, when the supermap fails, the output state remains the same as the initial state. Since the initial state is not altered on failure, one can reiterate this protocol to obtain an exponentially decreasing failure probability. Panel (a) represents the action of a success-or-draw supermap: when it succeeds, the target operation $f(U)$ is obtained; and when it draws, the identity operation id is obtained. Panel (b) illustrates a repeat-until-success protocol that is allowed for a success-or-draw protocol.

shows. In a usual probabilistic supermap, the input state is lost when it fails because an unknown quantum operation, which is the output quantum operation of the probabilistic supermap on failure, is applied on the input state, such as in the universal programmable quantum processor by port-based teleportation [20,21] and the probabilistic store and retrieve of unitary operations [11]. This fact, together with the impossibility to clone the quantum state, makes another trial not possible. However, while it is not possible to clone the quantum state, it is not known if it is possible to "keep" the quantum state when a probabilistic supermap fails. Thus, we propose a probabilistic supermap that "keeps" the quantum state on failure, or we call it a draw, as we are able to perform another trial when it happens as Fig. 1(b) shows. To summarize, a success-or-draw supermap has the following structure: when it is success, the target quantum operation is obtained; when it is draw, the identity operation is obtained; and the probability of success and draw sums up to 1.

Is it always possible to find a success-or-draw supermap for a given task? In Refs. [8,9], the probabilistic unitary inversion, which is a supermap transforming a unitary operation into its inverse, has been analyzed in the multiple calls scenario, and it is shown that the success-or-draw structure can be achieved by a construction of the quantum circuit. In this Letter, we show that the success-or-draw structure can be achieved for a larger class of supermaps by using a certain number of copies of an input quantum operation. Precisely, if there exists a probabilistic supermap transforming a single d-dimensional unitary operation into an arbitrary completely positive and trace-preserving (CPTP) map, then it is possible to construct a success-or-draw supermap with d copies of the input unitary operation. In particular, if a supermap is completely CP preserving (CCPP), a condition corresponding to complete positivity (CP) of quantum operations, it is probabilistically implementable [6]. Even if a supermap is not probabilistically implementable, if the supermap is linear, its approximated version, e.g., the structural physical approximation [23,24], is probabilistically implementable. Thus, this result applies to all linear supermaps on unitary operations if an approximation is also considered.

This result indicates that if there exists a probabilistic supermap transforming a unitary operation into a CPTP map, then the success probability of this supermap can approach 1 exponentially by allowing multiple calls of the input unitary operation. Moreover, the corresponding physical realization is given by repetitive trials of a single block of probabilistic supermap as shown in Fig. 1(b), and the cost of building the corresponding quantum circuit does not increase with the number of calls.

Success-or-draw supermap.—We first review the basics of supermaps. A supermap that uses the input quantum operations in a fixed order is known as a quantum comb, and is the one that can be implemented in the usual quantum circuit model. In Refs. [4,5], a formulation of a quantum comb is presented. To avoid confusion, we denote quantum operations with a tilde and supermaps with a double tilde. For example, given a unitary operator U, we denote the corresponding unitary operation by $\tilde{\mathcal{U}}$. A deterministic comb is described by a CCPP supermap with a set of linear constraints. A probabilistic comb, consisting of a success part and a failure part, can be described with two supermaps, say $\tilde{\tilde{\mathcal{S}}}$ and $\tilde{\tilde{\mathcal{F}}}$ respectively, which sum up to a deterministic comb.

Consider the probabilistic supermap transforming unitary operations $\{\tilde{\mathcal{U}}\}$ into CPTP maps $\{f(\tilde{\mathcal{U}})\}$. In the usual setting of a probabilistic supermap, this problem is formulated by the constraints

$$\tilde{\tilde{\mathcal{S}}}(\tilde{\mathcal{U}}) = p_U f(\tilde{\mathcal{U}}), \tag{1}$$

$$\tilde{\tilde{\mathcal{S}}}, \tilde{\tilde{\mathcal{F}}} \text{ is CCPP}, \tag{2}$$

$$\tilde{\tilde{\mathcal{S}}} + \tilde{\tilde{\mathcal{F}}} \text{ is a deterministic comb.} \tag{3}$$

For the success-or-draw supermap, the action on failure is also determined, and extra constraints on $\tilde{\tilde{\mathcal{F}}}$ are required. For convenience, in the following discussions we also assume that we have K calls to the input unitary operation $\tilde{\mathcal{U}}$. Since any unitary operation is transformed into the identity operation on failure, the corresponding constraints are given by

$$\tilde{\tilde{\mathcal{S}}}(\tilde{\mathcal{U}}^{\otimes K}) = p_U f(\tilde{\mathcal{U}}), \tag{4}$$

$$\tilde{\tilde{\mathcal{N}}}(\tilde{\mathcal{U}}^{\otimes K}) \propto \widetilde{id}, \tag{5}$$

$$\tilde{\tilde{\mathcal{S}}}, \tilde{\tilde{\mathcal{N}}} \text{ is CCPP}, \tag{6}$$

$$\tilde{\tilde{\mathcal{S}}} + \tilde{\tilde{\mathcal{N}}} \text{ is a deterministic comb,} \tag{7}$$

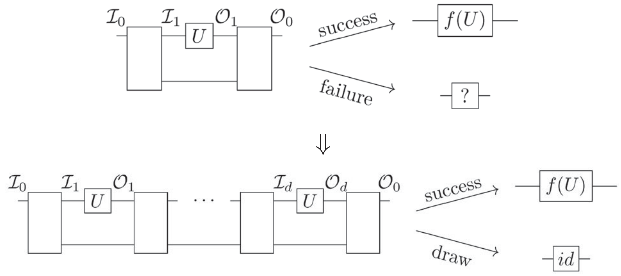

FIG. 2. A pictorial interpretation of Theorem 1. We consider the case in which there exists a probabilistic comb (upper) that transforms unitary operations U into CPTP maps $f(U)$, whose action is arbitrary on failure. Theorem 1 states that in this case, there exists a d-slot probabilistic comb (lower) that performs the same action on success, and performs the identity operation on failure/draw, which corresponds to the preservation of the input state.

where $\widetilde{id}$ denotes the identity operation, indicating that the input state does not change on failure. Here we use $\tilde{\tilde{\mathcal{N}}}$ instead of $\tilde{\tilde{\mathcal{F}}}$ to denote that it corresponds to draw instead of failure, and this condition is also known as the neutralization condition introduced in Ref. [13].

Main result.—Theorem 1 is the main result on the realizability of success-or-draw supermap. A pictorial interpretation of Theorem 1 is given by Fig. 2.

Theorem 1.—Given a probabilistic comb transforming d-dimensional unitary operations $\{\tilde{\mathcal{U}}\}$ to CPTP maps $\{f(\tilde{\mathcal{U}})\}$ as $\tilde{\tilde{\mathcal{S}}}_t{:}\tilde{\mathcal{U}} \mapsto p_U f(\tilde{\mathcal{U}})$. Then there exist $\varepsilon > 0$ and a set of probabilistic combs $\tilde{\tilde{\mathcal{S}}}$ and $\tilde{\tilde{\mathcal{N}}}$ that sum up to a deterministic comb, whose actions are given by

$$\tilde{\tilde{\mathcal{S}}}{:}\tilde{\mathcal{U}}^{\otimes d} \mapsto \varepsilon p_U f(\tilde{\mathcal{U}}), \tag{8}$$

$$\tilde{\tilde{\mathcal{N}}}{:}\tilde{\mathcal{U}}^{\otimes d} \mapsto (1-\varepsilon p_U)\widetilde{id}. \tag{9}$$

The proof of Theorem 1 is given in Supplemental Material [25], which includes Ref. [26]. The proof is constructive. We present a construction of the combs $\tilde{\tilde{\mathcal{S}}}$ and $\tilde{\tilde{\mathcal{N}}}$ from the comb $\tilde{\tilde{\mathcal{S}}}_t$; more precisely, we show a construction of S and N, the Choi operators [4,5,27,28] of $\tilde{\tilde{\mathcal{S}}}$ and $\tilde{\tilde{\mathcal{N}}}$, from S_t, the Choi operator of $\tilde{\tilde{\mathcal{S}}}_t$. The requirements for the combs are given by Eqs. (4)–(7), which need to be satisfied simultaneously.

While we only require that $S+N$ is a deterministic comb, that is, the input operations are used in a sequential way, the construction shown in the proof, found in Eq. (S43) of Supplemental Material [25], satisfies an extra condition

$$\mathrm{Tr}_{\mathcal{O}_0}(S+N) = \mathrm{Tr}_{\mathcal{O}_2\mathcal{O}_3\cdots\mathcal{O}_d\mathcal{O}_0}(S+N) \otimes \frac{I_{\mathcal{O}_2\mathcal{O}_3\cdots\mathcal{O}_d}}{d^{d-1}}, \tag{10}$$

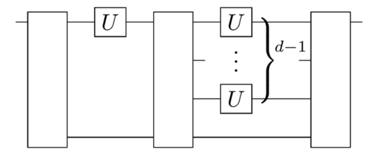

FIG. 3. The constructed success-or-draw comb in Supplemental Material [25] has an extra structure: it uses one copy of the unitary operation at first, and then uses the remaining $d-1$ copy of the unitary operation in parallel.

where $\mathcal{O}_0, \ldots, \mathcal{O}_d$ are the Hilbert spaces as shown in Fig. 2. This condition shows that the comb can be decomposed into two blocks as the quantum circuit shown in Fig. 3: the first block uses only a single unitary operation, while the second block uses the remaining $d-1$ unitary operations in parallel. Such a structure indicates that while the number of calls increases with d, the depth of this comb is constant at 2. Note that we can assume this structure if we consider only nonzero success probability, and in general, adding this assumption would decrease the success probability.

When the indefinite causal order [29–32] is considered, the construction can be replaced by a simpler one by exploiting the symmetry as Remark 2, and a higher success probability can be achieved in general.

Unitary inversion.—In this section, we analyze the probabilistic unitary inversion as a success-or-draw supermap. We consider only the two-dimensional case $d=2$ here. The optimal success probability can be obtained by the following semidefinite program (SDP):

$$\max\ p, \tag{11}$$

$$\text{s.t.}\quad \mathrm{Tr}_{\mathcal{I}\mathcal{O}}[S({J_{U_i}}^{\otimes K})^T] = pJ_{U_i^{-1}}, \tag{12}$$

$$\mathrm{Tr}_{\mathcal{I}\mathcal{O}}[N({J_{U_i}}^{\otimes K})^T] \le d^K J_{id}, \tag{13}$$

$$S \ge 0, \quad N \ge 0, \tag{14}$$

$$S+N \text{ is a deterministic comb}, \tag{15}$$

where S and N denote the Choi operators of the combs corresponding to success and draw, J_Λ denotes the Choi operator of a quantum operation Λ, and $\{U_i\}$ is a finite set of unitary operators that the corresponding Choi operators form a basis of the linear span of $\mathrm{span}\{J_U^{\otimes k}\}$ (see Refs. [8,9]).

For $K=2$, Theorem 1 indicates that the optimal success probability is positive as $p>0$. In fact, a numerical solution to this SDP shows that the optimal success probability is $p=1/3$, see Fig. 4. This problem is also considered in Ref. [8], where an explicit quantum circuit with the success-or-draw structure was presented, which has a success probability of $1/4$.

For comparison, we briefly state the protocol presented in Ref. [8]. The protocol is similar to the teleportation

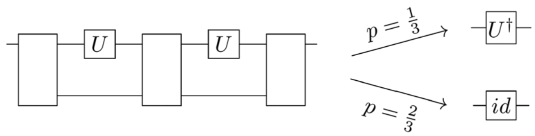

FIG. 4. The success-or-draw protocol for unitary inversion. When the unitary operation can be used twice, the optimal probability for success is $1/3$, whereas that of the optimal success-or-resetting protocol is $1/4$. In either case, the output on failure is the identity operation, which means the initial input state is preserved and it is possible to run the same protocol again with extra uses of the input unitary operation.

protocol, which generates the state $\sigma_x^i\sigma_z^j|\psi\rangle$ before correction, where $|\psi\rangle$ is the initial state and $(i,j)=(0,0)$, $(0,1),(1,0),(1,1)$ is the outcome of the Bell measurement. For the two-dimensional unitary inversion protocol presented in Ref. [8], we can obtain $U^{-1}\sigma_x^i\sigma_z^j|\psi\rangle$ with a single use of U by a small modification to the teleportation or gate teleportation protocol. This protocol successfully achieves unitary inversion when $(i,j)=(0,0)$. When it fails, on the other hand, we can obtain the state $\sigma_x^i\sigma_z^j|\psi\rangle$ by an extra use of U, and the input state $|\psi\rangle$ can be recovered by applying $(\sigma_x^i\sigma_z^j)^{-1}$, which achieves the neutralization supermap.

One difference between the optimal success-or-draw protocol we obtained and the protocol presented in Ref. [8] is that the latter is not only a success-or-draw protocol, but it also has another feature: it can be regarded as a success-or-resetting protocol. The latter protocol uses a single copy of a unitary operation to obtain its inverse, and when it fails, it results in a state that is "resettable" to be the input state by another unitary operation. Such a success-or-resetting protocol may have an advantage, as we can choose whether to continue the protocol by resetting after we know if it succeeded.

For $K=1$, we also prove that the optimal success probability is $p=0$, which means it is not possible to have a success-or-draw protocol. This result gives an explicit example that a success-or-draw protocol is not available. The proof is given in the Supplemental Material [25].

Discussion.—We have introduced a new structure for probabilistic supermap that we name success-or-draw structure. A probabilistic supermap with the success-or-draw structure can amplify its success probability by more calls of the input quantum operation in a sequential manner, which scales exponentially to 1 in the number of calls. A mathematical formulation for the success-or-draw supermap was presented. We considered the case in which the input quantum operation is a unitary operation, and we proved that any probabilistic supermap transforming unitary operations into CPTP maps can become a success-or-draw supermap by adding the number of copies of the unitary operation.

We then analyzed the problem of the two-dimensional unitary inversion. When two copies of an input unitary operation are allowed, Theorem 1 guarantees the existence of a nontrivial solution to this problem, and we also obtained the optimal solution numerically using SDP. A success-or-draw protocol for this problem was also presented previously in Ref. [8], and our numerical calculation shows that a higher success probability can be achieved if we only require the success-or-draw structure. We also proved that a success-or-draw protocol does not exist with a single copy of an input unitary operation.

Our result shows an advantage of probabilistic supermap, as it allows a success probability exponentially close to 1 in the sequential case. The number of calls is also undetermined for a success-or-draw protocol, and an average number of calls may become a suitable measure in practice. We also hope that the success-or-draw structure helps in a simpler physical implementation of supermaps.

This work was supported by MEXT Quantum Leap Flagship Program (MEXT Q-LEAP) Grants No. JPMXS0118069605 and No. JPMXS0120351339; Japan Society for the Promotion of Science (JSPS) by KAKENHI Grants No. 17H01694, No. 18H04286, and No. 18K13467; and Advanced Leading Graduate Course for Photon Science (ALPS). M. T. Q. also acknowledges the Austrian Science Fund (FWF) through the SFB project "BeyondC" (sub-project F7103), a grant from the Foundational Questions Institute (FQXi) as part of the Quantum Information Structure of Spacetime (QISS) Project (qiss.fr). The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 801110 and the Austrian Federal Ministry of Education, Science and Research (BMBWF). It reflects only the authors' view, the EU Agency is not responsible for any use that may be made of the information it contains.

[*]qingxiuxiong.dong@gmail.com
[†]marco.quintino@univie.ac.at
[‡]soeda@phys.s.u-tokyo.ac.jp
[§]murao@phys.s.u-tokyo.ac.jp

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, 2010).
[2] M. M. Wilde, *Quantum Information Theory* (Cambridge University Press, Cambridge, 2013).
[3] J. Watrous, Theory of quantum information, https://cs.uwaterloo.ca/~watrous/TQI-notes/ (2011).
[4] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Quantum Circuit Architecture, Phys. Rev. Lett. **101**, 060401 (2008).
[5] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Transforming quantum operations: quantum supermaps, Europhys. Lett. **83**, 30004 (2008).

[6] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Theoretical framework for quantum networks, Phys. Rev. A **80**, 022339 (2009).

[7] G. Chiribella, G. M. D'Ariano, and P. Perinotti, Optimal Cloning of Unitary Transformation, Phys. Rev. Lett. **101**, 180504 (2008).

[8] M. T. Quintino, Q. Dong, A. Shimbo, A. Soeda, and M. Murao, Reversing Unknown Quantum Transformations: A Universal Quantum Circuit for Inverting General Unitary Operations, Phys. Rev. Lett. **123**, 210502 (2019),

[9] M. T. Quintino, Q. Dong, A. Shimbo, A. Soeda, and M. Murao, Probabilistic exact universal quantum circuits for transforming unitary operations, Phys. Rev. A **100**, 062339 (2019).

[10] I. S. B. Sardharwalla, T. S. Cubitt, A. W. Harrow, and N. Linden, Universal refocusing of systematic quantum noise, arXiv:1602.07963.

[11] M. Sedlák, A. Bisio, and M. Ziman, Optimal Probabilistic Storage and Retrieval of Unitary Channels, Phys. Rev. Lett. **122**, 170502 (2019).

[12] G. Chiribella and D. Ebler, Optimal quantum networks and one-shot entropies, New J. Phys. **18**, 093053 (2016).

[13] Q. Dong, S. Nakayama, A. Soeda, and M. Murao, Controlled quantum operations and combs, and their applications to universal controllization of divisible unitary operations, arXiv:1911.01645.

[14] A. Bisio, G. Chiribella, G. M. D'Ariano, S. Facchini, and P. Perinotti, Optimal quantum learning of a unitary transformation, Phys. Rev. A **81**, 032324 (2010).

[15] I. L. Chuang and M. A. Nielsen, Prescription for experimental determination of the dynamics of a quantum black box, J. Mod. Opt. **44**, 2455 (1997).

[16] M. M. Wolf, Quantum Channels and Operations—Guided Tour, http://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf (2012).

[17] P. Busch, "No information without disturbance": quantum limitations of measurement, arXiv:0706.3526.

[18] W. K. Wootters and W. H. Zurek, A single quantum cannot be cloned, Nature (London) **299**, 802 (1982).

[19] Q. Dong, M. T. Quintino, A. Soeda, and M. Murao, Implementing positive maps with multiple copies of an input state, Phys. Rev. A **99**, 052352 (2019).

[20] S. Ishizaka and T. Hiroshima, Asymptotic Teleportation Scheme as aUniversal Programmable Quantum Processor, Phys. Rev. Lett. **101**, 240501 (2008).

[21] S. Ishizaka and T. Hiroshima, Quantum teleportation scheme by selecting one of multiple output ports, Phys. Rev. A **79**, 042306 (2009).

[22] J. Miyazaki, A. Soeda, and M. Murao, Complex conjugation supermap of unitary quantum maps and its universal implementation protocol, Phys. Rev. Research **1**, 013007 (2019).

[23] P. Horodecki, From limits of quantum operations to multicopy entanglement witnesses and state-spectrum estimation, Phys. Rev. A **68**, 052101 (2003).

[24] P. Horodecki and A. Ekert, Method for Direct Detection of Quantum Entanglement, Phys. Rev. Lett. **89**, 127902 (2002).

[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.150504 for the proofs of the main results.

[26] R. A. Bertlmann and P. Krammer, Bloch vectors for qudits, J. Phys. A Math. Gen. **41**, 235303 (2008).

[27] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, Rep. Math. Phys. **3**, 275 (1972).

[28] M.-D. Choi, Completely positive linear maps on complex matrices, Linear Algebra Appl. **10**, 285 (1975).

[29] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, Quantum computations without definite causal structure, Phys. Rev. A **88**, 022318 (2013).

[30] O. Oreshkov, F. Costa, and Č. Brukner, Quantum correlations with no causal order, Nat. Commun. **3**, 1092 (2012).

[31] M. Araújo, F. Costa, and Č. Brukner, Computational Advantage from Quantum-Controlled Ordering of Gates, Phys. Rev. Lett. **113**, 250402 (2014).

[32] M. Araújo, A. Feix, M. Navascués, and Č. Brukner, A purification postulate for quantum mechanics with indefinite causal order, Quantum **1**, 10 (2017).